Spectrum of Engineering Sciences
ISSN (e) 3007-3138 (p) 3007-312X
Volume 3, Issue 3, 2025

SCATTERING WIDTH OF THE SPHERICAL SHELL FILLED WITH DIELECTRIC MATERIAL IN NID SPACE

M Akbar[1], Mustansar Nadeem[2], Saeed Ahmed[*3]

[1]*Department Physics, Air University, Islamabad, Pakistan,*
[2]*Department Physics, FUUAST, Islamabad, Pakistan,*
Department Electronics, Quaid-i-Azam University, Islamabad, Pakistan,

[1]akbar5508126@yahoo.com, [2]mustansar.nadeem@fuuast.edu.pk, [3]saeedqau@gmail.com

DOI: https://doi.org/10.5281/zenodo.17199976

Keywords
Spherical shell, Dielectric Material, Fractional Dimension, Laplacian Differential Equation

Article History
Received on 20 January 2025
Accepted on 20 February 2025
Published on 27 February 2025

Corresponding Author: *
Saeed Ahmed

Abstract

In this paper, we have studied the effect on spherical shell of dielectric material analytically in fractional dimensional space. Here, the spherical shell has been considered as nanoparticle. The exact solution has been obtained in non integer dimensional space from the laplacian differential equation. This is a boundary value problem that is solved by separation variable method taking low frequency $\boldsymbol{\omega = 0}$ *Electric potential and cross section of the spherical shell is obtained in fractional dimensional space for the three regions, namely outside the spherical shell, between shell and hollow sphere and inside the sphere. Also the induced dipole moment is derived. This general solution reduces to the classical results by setting fractional parameter* $\boldsymbol{\alpha = 3}$ *which takes its value* $\boldsymbol{2 < \alpha \leq 3}$.

INTRODUCTION

The novel concept of fractional-dimensional space (FD space) is a much valuable in numerous discipline of physics discussed by several researchers [1-26]. This concept is also used to described the complexity of these structures. The solution of the electrostatic research problems [11-16], which have also been examined in the fractional dimensional space $\mathbf{2<\alpha\leq 3}$ .We have investigated the problem of a spherical shell of dielectric material [16] and have been solved in fractional dimensional space analytically [17]. The key emphasis is to utilze the Laplacian equation to track down the electric potential and generated dipole moment as well as cross section in fractional dimensional space. We have obtained a complete solutions for nanoparticle by using four boundary conditions while others [16-21] have used only results for various materials.

Materials and methods

We consider here a spherical shell filled with dielectric material as nanoparticle which is placed in fractional space. To study the cross section, we consider a spherical shell of inner radius a and the outer radius b. The inner core is made of material of permittivity and the spherical shell has been placed in a fractional space. We have to obtain the electric potential and the electric field produced due to the spherical shell filled with the dielectric material in NID Space, but most particularly in the cavity $\mathbf{r<a}$, as functions of permittivity. Thus the potential $\boldsymbol{\Psi}$ satisfies the Laplacian equation in FDS having fractional $\boldsymbol{\alpha}$-dimension in spherical polar coordinate systems which is defined as [16]

$$\left(\frac{d^2}{dr^2}+\frac{\alpha-1}{r}\frac{d}{dr}+\frac{1}{r^2}\left[\frac{d^2}{d\theta^2}+(\alpha-2)\cot\theta\frac{d}{d\theta}\right]-\right.$$
$$\left.\frac{1}{r^2\sin\theta}\left[\frac{d^2}{d\phi^2}+(\alpha-3)\cot\phi\frac{d}{d\phi}\right]\right)\psi=0$$

(1)

where the fractional parameter α lies in the range $\boldsymbol{2<\alpha\leq 3}$

In this case, the laplace equation for the potential independent of angle **ϕ** can be expressed as:

$$\nabla^2\Psi=\left(\frac{\partial^2}{\partial r^2}+\frac{\alpha-1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2\sin^{\alpha-2}\theta}\frac{\partial}{\partial\theta}\sin^{\alpha-2}\theta\frac{\partial}{\partial\theta}\right)\psi=0 \tag{2}$$

Eq(3) is separable and suppose

$$\Psi(r,\theta)=R(r)\Theta(\theta) \tag{3}$$

The obtained angular and radial differential equations [16] are

$$\left[\frac{d^2}{d\theta^2}+(\alpha-2)\cot\theta\frac{d}{d\theta}+l(l+\alpha-2)\right]\Theta(\theta)=0 \tag{4}$$

$$\left[\frac{d^2}{dr^2}+\frac{\alpha-1}{r}\frac{d}{dr}+\frac{l(l+\alpha-2)}{r^2}\right]R(r)=0 \tag{5}$$

Therefore, the solutions of **Ψ(r,θ)** in **α**-dimensional fractional space, that can expressed as [16]

$$\Psi(r,\theta)=\sum_{l=0}^{\infty}\left(a_l r^l+\frac{b_l}{r^{l+\alpha-2}}\right)C_l^{\frac{\alpha}{2}-1}(\cos\theta) \tag{6}$$

where a_l and b_l are unknown coefficients, which can be determined from the boundary conditions on **Ψ(r,θ).**

We construct here the solution for three different regions by satisfying the boundary conditions at **r=a** and **r=b**

For the outer region **r>b**, the potential must be of the form,

$$\Psi(r,\theta)=-E_0 r\cos\theta+\sum_{l=0}^{\infty}\frac{A_l}{r^{l+\alpha-2}}C_l^{\frac{\alpha}{2}-1}(\cos\theta) \tag{7}$$

where $\boldsymbol{H}=\boldsymbol{H_0}$ is the uniform field, at large distance.

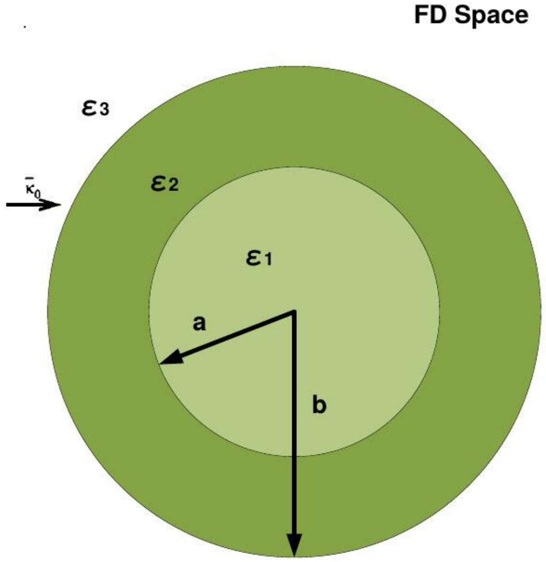

Fig. 1. Concentric Dielectric Sphere Filled with Dielectric Materials Placed in FD Space

For the inner regions $\boldsymbol{a<r}$, the potential can be written as

$$\Psi(r,\theta)=\sum_{l=0}^{\infty}\left(B_l r^l+\frac{C_l}{r^{l+\alpha-2}}\right)C_l^{\frac{\alpha}{2}-1}(\cos\theta) \tag{8}$$

For $r<a$,

$$\Psi(r,\theta)=\sum_{l=0}^{\infty}D_l r^l C_l^{\frac{\alpha}{2}-1}(\cos\theta) \tag{9}$$

All the coefficients with **l≠1** vanish. Then we can construct the solutions for different regions such as

$$\Psi_e(r,\theta)=\left[-H_0 r+Ar^{-(\alpha-1)}\right](\alpha-2)(\cos\theta),\ \ r>b \tag{10}$$

$$\Psi(r,\theta)=\left[Br+Cr^{-(\alpha-1)}\right](\alpha-2)(\cos\theta),\ \ a<r<b \tag{11}$$

$$\Psi(r,\theta)=Dr(\alpha-2)(\cos\theta),\ \ r<a. \tag{12}$$

The boundary conditions imposed at **r=a** and **r=b** such that $\mathbf{H_\theta}$ abd $\mathbf{B_r}$ be continuous. For **l=1**, the coefficients satisfy the four simultaneous equations.

$$\frac{\partial \Psi(r,\theta)}{\partial \theta}(b_-) = \frac{\partial \Psi(r,\theta)}{\partial \theta}(b_+) \tag{13}$$

$$\frac{\partial \Psi(r,\theta)}{\partial \theta}(a_-) = \frac{\partial \Psi(r,\theta)}{\partial \theta}(a_+) \tag{14}$$

$$\epsilon_2 \frac{\partial \Psi(r,\theta)}{\partial r}(b_-) = \epsilon_3 \frac{\partial \Psi(r,\theta)}{\partial r}(b_+) \tag{15}$$

and

$$\epsilon_1 \frac{\partial \Psi(r,\theta)}{\partial r}(a_-) = \epsilon_2 \frac{\partial \Psi(r,\theta)}{\partial r}(a_+) \tag{16}$$

By applying the boundary conditions, we determine the following four simplified equations

$$A - b^\alpha B - C = a_0 \tag{17}$$

where **α_0=E_0 b^α**

$$a_1 A + k_1 b^\alpha B - a_1 k_1 C = -a_0 \tag{18}$$

where α_1 =α-1, κ_2= $\varepsilon_2/\varepsilon_1$ and κ_1= $\varepsilon_2/\varepsilon_3$

$$a^\alpha B + C = a^\alpha D \tag{19}$$

$$a^\alpha \kappa_2 B - a_1 \kappa_2 C = a^\alpha D \tag{20}$$

Solving Eq.(17) and Eq.(18), we find

$$C = -\frac{(\kappa_1 + a_1)}{a_1(1-\kappa_1)} b^\alpha B - \frac{a_0 \alpha}{a_1(1-\kappa_1)} \tag{21}$$

Solving Eq.(19) and Eq.(20), we obtain

$$C = \frac{(\kappa_2 - 1)}{(a_1 \kappa_2 + 1)} a^\alpha B \tag{22}$$

Now by comparing Eq.(21) and Eq.(22), we get

$$B = -\frac{a_0 \alpha (\kappa_2 a_1 + 1)}{a_1 a^\alpha (\kappa_2 - 1)(1-\kappa_1) + (\kappa_1 + a_1)(\kappa_2 a_1 + 1) b^\alpha} \tag{23}$$

Similarly

$$C = -\frac{a_0 \alpha a^\alpha (\kappa_2 - 1)}{a_1 a^\alpha (\kappa_2 - 1)(1-\kappa_1) + (\kappa_1 + a_1)(\kappa_2 a_1 + 1) b^\alpha} \tag{24}$$

From Eq.(19) we find

$$D = -\frac{a_0 \alpha^2 \kappa_2}{a_1 a^\alpha (\kappa_2 - 1)(1-\kappa_1) + (\kappa_1 + a_1)(\kappa_2 a_1 + 1) b^\alpha} \tag{25}$$

Solving for Constant A, by putting the value of unknown coefficients B and C in Eq.(18), we find the simplified coefficient A:

$$A = a_0 \frac{(\kappa_2 a_1 + 1)(\kappa_1 - 1) b^\alpha + (\kappa_1 a_1 + 1)(\kappa_2 - 1) a^\alpha}{a_1 a^\alpha (\kappa_2 - 1)(1-\kappa_1) + (\kappa_1 + a_1)(\kappa_2 a_1 + 1) b^\alpha} \tag{26}$$

where **α_1 =α-1and κ_2= $\varepsilon_2/\varepsilon_1$**

By simplification, we obtain

$$A = E_0 b^\alpha \frac{(\epsilon_2 - \epsilon_3)(\epsilon_1 + (\alpha-1)\epsilon_2) + (\epsilon_1 - \epsilon_3)(\epsilon_3 + (\alpha-1)\epsilon_2)(\frac{a}{b})^\alpha}{(\epsilon_2 + (\alpha-1)\epsilon_3)(\epsilon_1 + (\alpha-1)\epsilon_2) + (\alpha-1)(\epsilon_1 - \epsilon_2)(\epsilon_2 - \epsilon_3)(\frac{a}{b})^\alpha} \tag{27}$$

$$B = E_0 \alpha \frac{(\epsilon_1 + (\alpha-1)\epsilon_2)\epsilon_3}{(\epsilon_2 + (\alpha-1)\epsilon_3)(\epsilon_1 + (\alpha-1)\epsilon_2) + (\alpha-1)(\epsilon_1 - \epsilon_2)(\epsilon_2 - \epsilon_3)(\frac{a}{b})^\alpha} \tag{28}$$

$$C = -E_0 \alpha a^\alpha \frac{(\epsilon_2 - \epsilon_1)\epsilon_3}{(\epsilon_2 + (\alpha-1)\epsilon_3)(\epsilon_1 + (\alpha-1)\epsilon_2) + (\alpha-1)(\epsilon_1 - \epsilon_2)(\epsilon_2 - \epsilon_3)(\frac{a}{b})^\alpha} \tag{29}$$

$$D = -E_0 \frac{\alpha^2 \epsilon_3 \epsilon_1}{(\epsilon_2 + (\alpha-1)\epsilon_3)(\epsilon_1 + (\alpha-1)\epsilon_2) + (\alpha-1)(\epsilon_1 - \epsilon_2)(\epsilon_2 - \epsilon_3)(\frac{a}{b})^\alpha} \tag{30}$$

Now we easily find the potential for the following three regions:

$$\Psi_e(r,\theta) = E_0\left[-1 + \frac{(\epsilon_2 - \epsilon_3)(\epsilon_1 + (\alpha-1)\epsilon_2) + (\epsilon_1 - \epsilon_3)(\epsilon_3 + (\alpha-1)\epsilon_2)(\frac{a}{b})^\alpha}{(\epsilon_2 + (\alpha-1)\epsilon_3)(\epsilon_1 + (\alpha-1)\epsilon_2) + (\alpha-1)(\epsilon_1 - \epsilon_2)(\epsilon_2 - \epsilon_3)(\frac{a}{b})^\alpha}(\frac{b}{r})^\alpha\right] r(\alpha - -2)(\cos\theta), \quad r > b \tag{31}$$

$$\Psi_i(r,\theta) = -E_0 \alpha \left[\frac{(\in_2 - \in_1) + (\in_1 + (\alpha-1)\in_2)}{(\in_2 + (\alpha-1)\in_3)(\in_1 + (\alpha-1)\in_2) + (\alpha-1)(\in_1 - \in_2)(\in_2 - \in_3)\left(\frac{a}{b}\right)^\alpha}\left(\frac{a}{r}\right)^\alpha\right]\{\in_3 r(\alpha-2)(\cos\theta)\}, a < r < b \tag{32}$$

$$\Psi_c(r,\theta) = -E_0 \frac{\alpha^2 \epsilon_3 \epsilon_1}{(\epsilon_2 + (\alpha-1)\epsilon_3)(\epsilon_1 + (\alpha-1)\epsilon_2) + (\alpha-1)(\epsilon_1 - \epsilon_2)(\epsilon_2 - \epsilon_3)(\frac{a}{b})^\alpha} r(\alpha - 2)(\cos\theta), \quad r < a. \tag{33}$$

When the size of the core-shell (nanoparticle) is very small as compared to the wavelength, then the polarizability can be attained from the amplitude of the dipole component in the external domain shown in Eq. (31) In other words, the polarizability is the proportionality factor of dipole moment given by [16-17]

$$\Psi_e(r,\theta)=E_0\left[\frac{(\in_2-\in_3)(\in_1+(\alpha-1)\in_2)+(\in_1-\in_3)}{(\in_2+(\alpha-1)\in_3)(\in_1+(\alpha-1)\in_2)+(\alpha-1)}\right.$$

$$\left.\frac{(\in_3+(\alpha-1)\in_2}{(\in_1-\in_2)(\in_2-\in_3)(\frac{a}{b})^{\alpha}}(\frac{b}{r})^{\alpha}\right]r(\alpha-2)\left(\mathrm{Cos}\,\theta\right),r>b$$

(34)

Where $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$ aand $\boldsymbol{\varepsilon}_3$ are named as the permittivity of core, shell and the host medium, respectively. By choosing **α=3**, it reduces **the Eqs** (31), (32), (33) to ordinary 3D space [18] forms. When a uniform electromagnetic field hits the core-shell, the dipole moment induces on its surface The potential $\boldsymbol{\Psi}_{\mathrm{dip}}$ for an ideal dipole at a distance r in an unbounded medium can be expressed as [20]

$$\Psi_{dip}=\frac{p.r}{4\pi\epsilon_3 r^3}=\frac{p\cos\theta}{4\pi\epsilon_3 r^2} \qquad (35)$$

here **p** means the dipole moment, that induces due to the electric polarizability of the core-shell (nanoparticle), can be shown as as [20]

$$p=4\pi\epsilon_3E_0b^{\alpha}\left[\frac{(\epsilon_2-\epsilon_3)(\epsilon_1+(\alpha-1)\epsilon_2)+(\epsilon_1-\epsilon_3)(\epsilon_3+(\alpha-1)\epsilon_2)(\frac{a}{b})^{\alpha}}{(\epsilon_2+(\alpha-1)\epsilon_3)(\epsilon_1+(\alpha-1)\epsilon_2)+(\alpha-1)(\epsilon_1-\epsilon_2)(\epsilon_2-\epsilon_3)(\frac{a}{b})^{\alpha}}\right] \qquad (36)$$

The dipole's potential can be defined as follows

$$\Psi_{dip}=E_0\left[\frac{(\epsilon_2-\epsilon_3)(\epsilon_1+(\alpha-1)\epsilon_2)+(\epsilon_1-\epsilon_3)(\epsilon_3+(\alpha-1)\epsilon_2)(\frac{a}{b})^{\alpha}}{(\epsilon_2+(\alpha-1)\epsilon_3)(\epsilon_1+(\alpha-1)\epsilon_2)+(\alpha-1)(\epsilon_1-\epsilon_2)(\epsilon_2-\epsilon_3)(\frac{a}{b})^{\alpha}}(\frac{b}{r})^{\alpha}\right]r(\alpha-2)(\cos\theta) \qquad (37)$$

The dipole's moment p due to the electric polarizability of the core-shell [20] and it may be specified by the polarizability P_0 ,which is related to **p** as

$$p=\epsilon_3P_0E_0 \qquad (38)$$

The dipole term shown in terms of potential **b** corresponding to the host medium, and presents the polarizability $\mathbf{P}_0$ of the spherical shell, that can be written as [19]

$$P_0=4\pi r^3\left[\frac{(\epsilon_2-\epsilon_3)(\epsilon_1+(\alpha-1)\epsilon_2)+(\epsilon_1-\epsilon_3)(\epsilon_3+(\alpha-1)\epsilon_2)(\frac{a}{b})^{\alpha}}{(\epsilon_2+(\alpha-1)\epsilon_3)(\epsilon_1+(\alpha-1)\epsilon_2)+(\alpha-1)(\epsilon_1-\epsilon_2)(\epsilon_2-\epsilon_3)(\frac{a}{b})^{\alpha}}\right] \qquad (39)$$

The scattering cross-section $\boldsymbol{\sigma}_{\mathrm{sca}}$ is directly associated to the squared extent of polarizability, and consequently, it may be expressed as

$$\sigma_{sca}=\frac{k^4}{6\pi(\epsilon_3)^2}|P_0|^2 \qquad (40)$$

Moreover, the absorption cross-section $\boldsymbol{\sigma}_{\mathrm{abs}}$ is comparative to the imaginary part (Im) of polarizability, that is expressed as

$$\sigma_{abs}=\frac{k}{\epsilon_3}Im(P_0) \qquad (41)$$

1. RESULTS AND DISCUSSION

We have derived an exact solution by solving Laplacian equation in fractional space taking wave number **k**=0. Here, we have investigated the effect on spherical shell of dielectric material analytically in fractional dimensional space. This boundary value problem helps us to determine the electric potential and cross section of the spherical shell (nanoparticle) in fractional dimensional space for the three regions, namely outside the spherical

shell, between shell and core shell Also, the induced dipole moment is derived. In this article we have calculated a complete mathematical solution while all others [17-21] have used its final results Therefore, our solution to have scattering width in low frequency makes it more general exact solution. This is a general solution that can be employed to obtain a solution for various materials for the three regions of the dielectric shell.

2. CONCLUSION

We have investigated a close form solution in non integer dimensional space for, spherical shell (nanoparticle) and core-shell filled with dielectric material. Fractional dynamics plays a vital role in illustrating the intricative phenomena. The core-shell nanoparticles were considered to have dimensions smaller than the operating wavelength. Under such conditions, the analyses of polarizability and the scattering and absorption cross-sections. It has been found that a varying FD parameter causes the increase of FD space parameter generally results in the resonance peaks.This general solution can be applied for different materials in side and outside the spherical shell. For all the investigated cases when $\boldsymbol{\alpha}=\mathbf{3}$ the classical results are retrieved.

3. CONFLICT OF INTEREST

Authers have no conflict of interest.

REFERENCES

C.G. Bollini, and J.J. Giambiagi. Dimensional renormalization: The number of dimensions as a regularizing parameter. *Nuovo Cimento B* 12 (1972) 2026.

J.F. Ashmore. On renormalization and complex spacetime dimensions. *Communications in Mathematical Physics* 29: 177-187 (1973).

K.G. Wilson. Quantum field-theory models in less than 4 dimension. *Physical Review D* 7(10): 2911-2926 (1973).

F.H. Stillinger. Axiomatic basis for spaces with noninteger dimension. *Journal of Mathematical Physics* 18(6): 1224-1234 (1977).

C.M. Bender, and S. Boettcher. Dimensional expansion for the Ising limit of quantum field theory. *Physical Review D* 48(10): 4919-4923 (1993).

C.M. Bender, and K.A. Milton. Scalar Casimir effect for a D-dimension sphere. *Physical Review D* 50 (10): 6547-6555 (1994).

V.E. Tarasov. Fractional generalization of Liouville equations. *Chaos: An Interdisciplinary Journal of Nonlinear Science* 14(1): 123-127 (2004).

V.E. Tarasov. Electromagnetic fields on fractals. *Modern Physics Letters A* 21(20): 1587-1600 (2006).

A. Zeilinger, and K. Svozil. Measuring the dimension of spacetime. *Physical Review Letters* 54(24): 2553-2555 (1985).

S. Muslih, and D. Baleanu. Fractional multipoles in fractional space. *Nonlinear Analysis: Real World Applications* 8(1): 198-203 (2007).

C. Palmer, and P.N. Stavrinou. Equations of motion in a noninteger-dimensionspace. *Journal of Physics A: Mathematical and General* 37(27): 6987-7003 (2004).

J.D. Jackson. Classical Electrodynamics. 3rd Ed. *John Wiley, New York, USA,* pp. 201-202 (1999).

J.A. Stratton, Electromagnetic Theory. *Wiley-IEEE Press,* (1941).

T. Myint-U, and L. Debnath (4th ed). Linear Partial Differential Equations for Scientists and Engineers. *Springer Science & Business Media* (2007).

V.E. Tarasov, Gravitational field of fractals distribution of particles *Celestial Mechanics and Dynamical Astronomy* 94: 1-115 (2006).

D. Baleanu, A.K. Golmankhaneh, and A.K. Golmankhaneh. On electromagnetic field in fractional space. *Nonlinear Analysis: Real World Applications* 11(1): 288-292 (2010).

X.F. He. Excitons in anisotropic solids: The model of fractional-dimensional space. *Physical. Review B* 43(3): 2063-2069 (1991).

Sihvola A.H. Character of surface plasmons in layered spherical structures. *Progress. In Electromagnetics. Research.* 62:317–331 2006.

Kemp B.A., Rahman M.H. Analytical model of plasmonic resonance from multiple core-shell nanoparticles. *Opt. Eng.* 56 **2017;:121903.**

Bohren C.F., Huffman D.R.*Absorption and Scattering of Light by Small Particles.*Wiley; New York, NY, USA: 2008.

A. Ali, M. A. Ashraf, Q.A., Baqir, M.A.; Choudhury, P.K. On the Core-Shell Nanoparticle in Fractional Dimensional Space. *Materials* 2020

V.E. Tarasov. Vector Calculus in Non-Integer Dimensional Space and its Applications to Fractal Media *Communications in Nonlinear Science and Numerical Simulation* 20(2): 360-374 (2015).

S. Ahmed, M. Akbar and M.I. Shahzad. Response of Homogeneous Conducting Sphere in Non-Integer Dimensional Space. *Proceedings of the Pakistan Academy of Sciences: A. Physical and Computational Sciences* 58(4): 11-16 (2021).

M.I. Shahzad, M. Akbar and S. Ahmed. Mathematical Analysis on Conducting Sphere Embedded in Non-Integer Dimensional Space. *Proceedings of the Pakistan Academy of Sciences: Part A. Physical and Computational Sciences* 59(1): 83-87 (2022).

S. Ahmed, M. Akbar and M.I. Shahzad. Close Form Solution for Dielectric Cylindrical Shell in Fractional Dimensional Space. *Proceedings of the Pakistan Academy of Sciences: Part A (Physical and Computational Sciences).* 59(2): 15-18 (2022).

M. Akbar, M.I. Shahzad and S. Ahmed. Evaluation of Electric Field for a Dielectric Cylinder Placed in Fractional Space. *Proceedings of the Pakistan Academy of Sciences: Part A (Physical and Computational Sciences)* 59(2): 19-22 (2022)