

NUMERICAL APPROXIMATION OF RIEMANN PROBLEMS FOR A LIQUID-GAS TWO-PHASE ISENTROPIC FLOW MODEL: COMPUTATIONAL ASSESSMENT OF THREE EXPLICIT NUMERICAL SCHEMES

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DOI: <https://doi.org/10.5281/zenodo.23260945>

Keywords

Riemann problem; two-phase flow; hyperbolic conservation laws; finite-volume methods; numerical simulation.

Article History

Received: 01 January 2026

Accepted: 16 March 2026

Published: 31 March 2026

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Abstract

This paper revisits the numerical solution of Riemann problems for a one-dimensional liquid-gas two-phase isentropic flow model. The main purpose is to provide a corrected and reproducible comparison of three classical finite-volume schemes: Lax-Friedrichs, two-step Lax-Wendroff, and FORCE. The numerical solutions are compared with a model-specific Riemann reference, using the same mesh, CFL condition, boundary treatment, and admissibility checks for all three methods. For the baseline parameters $k = 0.6$ and $\gamma = 0.8$, the first test problem contains two rarefaction waves separated by a contact discontinuity, whereas the second contains two shocks with a contact in between. At $N = 400$, Lax-Wendroff gives the smallest integral errors in both test problems, although small oscillations appear near discontinuities. Lax-Friedrichs is more diffusive, while FORCE gives intermediate errors and wave resolution. A separate smooth translating-contact test gives observed convergence orders of 0.991, 2.000, and 0.997 for Lax-Friedrichs, Lax-Wendroff, and FORCE, respectively. Additional tests involving left-moving waves, mixed wave patterns, and positive low-density states were also carried out. Across the benchmark calculations, no non-positive accepted phase masses were observed, and the conservation residual remained close to machine precision. These results provide a verified numerical baseline for further work on higher-resolution methods for this reduced two-phase model.

1. INTRODUCTION

Two-phase flow appears in many engineering applications, including pipeline transport, petroleum production, chemical processing, and energy systems. From a mathematical point of view, these problems are difficult because even relatively simple reduced models can form nonlinear hyperbolic systems. Their solutions

may contain shocks, rarefaction waves, and contact discontinuities, so the numerical method must preserve conservation while still resolving different wave families with reasonable accuracy.

Drift-flux modelling has a long history, including the classical treatment of average phase concentration by Zuber and Findlay (1965). A wide range of numerical approaches has since

been developed for two-phase and drift-flux models. Earlier studies include the linearized two-fluid Riemann solver of Toumi and Kumbaro (1996), the slip-relation model considered by Romate (1998), the hybrid primitive-conservative schemes of Fjelde and Karlsen (2002), and the centred drift-flux methods studied by Munkejord et al. (2006). Roe-type ideas were also applied to drift-flux systems by Flåtten and Munkejord (2006), while later studies developed model-specific Riemann solvers and examined wave interactions in related systems (Kuila et al., 2015, 2016; Minhajul et al., 2018). These works make one point especially clear: numerical wave profiles should be interpreted using the mathematical structure of the model, because a computed transition alone does not identify whether the underlying wave is shock, rarefaction, or contact discontinuity.

More recent work has focused on improving resolution near discontinuities. Ouffa et al. (2021) studied a weighted average flux approach for an isentropic drift-flux model, Rehman and Meraj (2022) considered a THINC-BVD formulation with an AUSMV-type flux, and Nascimento and Rosa (2024) investigated localized artificial diffusivity for one-dimensional drift-flux equations. These developments show that the performance of a numerical method depends not only on the underlying flux, but also on reconstruction, limiting, time integration, and the model being solved.

The model considered in the present paper is the reduced liquid-gas two-phase isentropic system studied in the author's MS thesis and later reported in preprint form (Rab, 2023, 2025). It contains conservation equations for the two phase masses together with a liquid-momentum equation, while pressure is prescribed as a nonlinear function of the total phase mass. Related analytical studies by Shen (2020a, 2020b) provide useful background for understanding the Riemann structure of this class of models. In the formulation considered here, the system has two genuinely nonlinear acoustic families and one linearly degenerate middle family associated with a phase-composition contact.

During preparation of the present article, the original mathematical and numerical formulation was rechecked in detail. This revealed several points that needed correction before the earlier results could be used reliably. In particular, the characteristic polynomial must distinguish correctly between the spectral variable and the pressure exponent, and the first and third characteristic fields are genuinely nonlinear, while the middle field is linearly degenerate. The classification of the waves also needs to come from the Riemann solution itself. A shock that appears smooth after numerical diffusion is still a shock; its visual appearance does not change its mathematical type.

The numerical implementation required similar care. The CFL estimate must include both positive and negative characteristic speeds. The time step used to update the conserved state must also be the one used to advance the numerical clock, and the final step should end exactly at the required final time. The computational grid should contain exactly N finite-volume cell averages with $\Delta x = 2/N$ on $[-1, 1]$. Boundary conditions should be defined explicitly, and the plotted velocity must be reconstructed as $u = j/n$, rather than confusing the momentum variable $j = nu$ with velocity itself. These details directly affect reproducibility and the interpretation of the computed results.

For this reason, the objective of the present study is intentionally limited. No new numerical flux is proposed. Instead, three classical schemes—Lax-Friedrichs, two-step Lax-Wendroff, and FORCE—are compared within the same corrected finite-volume framework. All three methods use the same mesh, CFL condition, boundary treatment, final-time control, and admissibility checks. Their numerical solutions are assessed against a model-specific Riemann reference using quantitative errors and mesh refinement in addition to visual comparisons.

The comparison is also extended beyond the two original Riemann problems. A smooth translating-contact problem is used to check the expected convergence behavior of the implementations. A separate parameter study compares the baseline value $\gamma = 0.8$ with $\gamma = 1.4$

and shows that the pressure exponent can change the wave pattern. Additional tests include left-moving waves, a stationary contact, a mixed shock-rarefaction configuration, and positive low-density states. Conservation, Rankine-Hugoniot jump residuals, rarefaction residuals, pressure-root residuals, and quadrature checks are used as further verification of both the finite-volume solver and the reference construction.

The numerical results show a clear but qualified pattern. Lax-Wendroff gives the smallest measured integral errors for the two principal $N = 400$ tests and shows second-order behavior for the smooth contact problem, although oscillations remain visible near discontinuities. Lax-Friedrichs is the most diffusive of the three methods, while FORCE gives intermediate behavior. These results should not be interpreted as a universal ranking; rather, they illustrate the trade-off between numerical diffusion, dispersion, and wave resolution for the present model and parameter choices.

The rest of the paper is organized as follows. Section 2 reviews the related analytical and numerical literature. Section 3 presents the mathematical model and its Riemann structure. Section 4 describes the numerical methods and reference solution. Section 5 discusses the numerical results, convergence study, parameter sensitivity, and verification tests. The main conclusions and possible extensions are given in Section 6.

2. Literature Review

The numerical treatment of two-phase flow models is closely connected with the structure of their characteristic waves. For this reason, a substantial part of the literature has focused on approximate Riemann solvers and conservative finite-volume methods. Toumi and Kumbaro (1996) developed an approximate linearized Riemann solver for a two-fluid model, while Romate (1998) considered a two-phase model in which the slip relation is supplied numerically. Although these formulations differ from the reduced model studied here, they provide useful background for understanding how phase

interactions and multiple wave families are handled within conservative numerical methods. For drift-flux models, several authors have proposed methods that balance characteristic information with numerical dissipation. Fjelde and Karlsen (2002) developed high-resolution hybrid primitive-conservative upwind schemes, whereas Munkejord et al. (2006) investigated a multistage centred approach. The latter is relevant to the present work because it demonstrates that useful approximations for two-phase flow can be obtained without solving a full Riemann problem at every interface. Flåtten and Munkejord (2006) examined a Roe-type approximate solver for a drift-flux system, providing another example of how characteristic information can be incorporated into the numerical flux.

Model-specific Riemann solvers have also received considerable attention. Kuila et al. (2015) developed a Riemann solver for a compressible two-phase model, and Kuila et al. (2016) considered the Riemann problem for drift-flux equations. Minhajul et al. (2018) studied wave interactions within a drift-flux framework, while Zeidan et al. (2023) investigated a related Riemann problem with a modified Chaplygin-type closure. The governing equations and closures in these studies are not identical to those considered here, but they highlight an important methodological point: numerical results should be interpreted with respect to the particular model being solved. A wave pattern obtained for one closure cannot automatically be transferred to another.

Analytical studies of the Riemann problem are equally useful for numerical verification. Shen (2020a) examined asymptotic limits of Riemann solutions for an isentropic drift-flux model, while Shen (2020b) studied singular limits for a liquid-gas two-phase isentropic model closely related to the present system. Such results provide a mathematical basis for distinguishing shocks, rarefaction fans, and contact discontinuities and help avoid interpreting a numerically smeared discontinuity as a different physical wave simply because its computed profile appears smooth.

The three methods considered in this paper have different numerical characteristics. Lax-Friedrichs is a classical first-order centred method and is known for relatively strong numerical diffusion. The two-step Lax-Wendroff method has higher accuracy in smooth regions but may generate dispersive oscillations close to discontinuities. FORCE combines Lax-Friedrichs and Richtmyer-type contributions and therefore provides a useful intermediate centred flux. Broader analyses of centred and FORCE-type schemes can be found in Toro and Billett (2000) and Toro et al. (2009), while the general finite-volume and Riemann-solver framework is discussed by LeVeque (2002) and Toro (2009).

More recent work has aimed to improve discontinuity resolution while limiting spurious oscillations. Ouffa et al. (2021) applied a weighted average flux method to an isentropic drift-flux model. Rehman and Meraj (2022) studied a THINC-BVD formulation, and Nascimento and Rosa (2024) considered localized artificial diffusivity for one-dimensional drift-flux equations. These studies show that current high-resolution methods go well beyond the three classical schemes used in the present comparison. For that reason, the contribution of this paper is not presented as a new state-of-the-art numerical method. Instead, the purpose is to establish a carefully checked baseline for the specific reduced liquid-gas model considered here.

The present study is therefore positioned as a verification and benchmarking exercise. It combines classical numerical fluxes with a model-specific Riemann reference, cell-average error measurements, a smooth-contact convergence test, parameter sensitivity, admissibility monitoring, conservation accounting, and independent residual checks. This framework provides a reliable point of comparison for later higher-resolution extensions, where any claimed improvement can be assessed against a baseline whose mathematical wave structure and numerical implementation have already been examined carefully.

3. Mathematical Model and Riemann Structure

3.1 Governing equations

We consider the one-dimensional liquid-gas two-phase isentropic model

$$\begin{aligned} m_t + (mu)_x &= 0, \\ n_t + (nu)_x &= 0, \\ (nu)_t + [nu^2 + k(m+n)^\gamma]_x &= 0. \end{aligned}$$

Here, m and n represent the phase mass densities, while u is the common velocity associated with the liquid-momentum variable. The pressure is prescribed through

$$p = k(m+n)^\gamma,$$

with $k > 0$ and $\gamma > 0$. Introducing $j = nu$ and $S = m + n$, the conservative state and flux are

$$Q = \begin{pmatrix} m \\ n \\ j \end{pmatrix}, \quad F(Q) = \begin{pmatrix} mj/n \\ j \\ j^2/n + kS^\gamma \end{pmatrix}.$$

The numerical calculations in this study are restricted to states with positive phase masses, $m > 0$ and $n > 0$. These inequalities define the computationally admissible set used in the simulations. The present work does not independently establish the physical calibration or dimensional interpretation of the pressure closure, so the conclusions should be understood in relation to the mathematical model written above.

3.2 Characteristic structure

For primitive variables $W = (m, n, u)^T$, the system can be written

$$W_t + B(W)W_x = 0,$$

where

$$B(W) = \begin{pmatrix} u & 0 & m \\ 0 & u & n \\ a/n & a/n & u \end{pmatrix}, \quad a = k\gamma(m+n)^{\gamma-1}.$$

The characteristic polynomial is

$$(\lambda - u)[n(\lambda - u)^2 - k\gamma(m+n)^\gamma] = 0.$$

Hence the acoustic speed satisfies

$$c^2 = \frac{k\gamma(m+n)^\gamma}{n}$$

and the three characteristic speeds are

$$\lambda_1 = u - c, \quad \lambda_2 = u, \quad \lambda_3 = u + c.$$

For positive phase masses and positive model parameters, the acoustic speed is positive. The three eigenvalues are therefore distinct, so the

system is strictly hyperbolic on the admissible domain considered in this work.

A convenient choice of right eigenvectors in primitive variables is

$$r_1 = (m, n, -c)^T, \quad r_2 = (1, -1, 0)^T, \quad r_3 = (m, n, c)^T.$$

The corresponding directional derivatives are

$$\begin{aligned} \nabla \lambda_1 \cdot r_1 &= -\frac{\gamma+1}{2}c, & \nabla \lambda_2 \cdot r_2 &= 0, \\ \nabla \lambda_3 \cdot r_3 &= \frac{\gamma+1}{2}c. \end{aligned}$$

It follows that the first and third characteristic fields are genuinely nonlinear, while the second field is linearly degenerate. This distinction determines the basic Riemann structure: the outer families may produce shocks or rarefaction waves, whereas the middle family carries a contact discontinuity.

3.3 Contact discontinuity and acoustic wave curves

Along the middle eigenvector, $dm = -dn$ and $du = 0$. Therefore $S = m + n$ is constant, and so is $p = kS^\gamma$. Across an exact contact,

$$u_L = u_R, \quad p_L = p_R, \quad S_L = S_R,$$

although the individual phase masses m and n may change. Contact can therefore be clearly visible in the phase-mass profiles even though there is no jump in velocity or pressure.

For the genuinely nonlinear families, the differential relations give

$$\frac{dm}{m} = \frac{dn}{n},$$

so the phase ratio $r = m/n$ remains constant. Hence $m = rn$, $S = (1+r)n$, and the pressure can be expressed as $p = An^\gamma$ with $A = k(1+r)^\gamma$. The acoustic integral curves satisfy

$$du = \mp c \, d(\ln n),$$

with the upper and lower signs associated with the first and third families. For $\gamma \neq 1$, the corresponding Riemann invariants can be written as $u + 2c/(\gamma-1)$ for the first family and $u - 2c/(\gamma-1)$ for the third.

Across discontinuity moving with speed s , the Rankine-Hugoniot conditions are

$$\begin{aligned} s[m] &= [mu], & s[n] &= [nu], & s[nu] &= [nu^2 + p]. \end{aligned}$$

For a non-contact acoustic shock, the first two jump conditions imply that the ratio m/n remains unchanged across the shock. The remaining relation can be written as

$$(u_R - u_L)^2 = (p_R - p_L) \left(\frac{1}{n_L} - \frac{1}{n_R} \right),$$

The physically admissible compressive branch is then selected using the corresponding Lax inequalities.

3.4 Riemann problem

The Riemann problem is defined by piecewise constant initial data

$$W(x, 0) = \begin{cases} W_L, & x < 0, \\ W_R, & x > 0, \end{cases}$$

the entropy solution is self-similar in $\xi = x/t$ and consists of a first-family acoustic wave, a middle contact, and a third-family acoustic wave. The acoustic waves are classified by the star pressure p_* . A side produces a shock if $p_* > p_s$ and a rarefaction if $p_* < p_s$. This criterion is used throughout the study; wave type is never inferred from the visual thickness of a numerical transition.

4. Methodology

4.1 Common finite-volume framework

The computational interval is $[-1, 1]$ and is divided into exactly N uniform finite-volume cells with $\Delta x = 2/N$. Let Q_i^n denote the cell average in cell i at time t^n . Each method is written in conservative flux form,

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t^n}{\Delta x} (\hat{F}_{i+1/2}^n - \hat{F}_{i-1/2}^n).$$

To make the comparison fair, the same mesh, initial data, CFL rule, boundary treatment, final-time procedure, admissibility checks, and error measures are used for all three schemes. Otherwise, a difference caused by implementation details could be mistaken for a difference between the numerical fluxes themselves.

4.2 Lax-Friedrichs, Lax-Wendroff, and FORCE fluxes

For the Lax-Friedrichs method, the interface flux is

$$\hat{F}_{i+1/2}^{LF} = \frac{1}{2} [F(Q_i) + F(Q_{i+1})] - \frac{\Delta x}{2\Delta t} (Q_{i+1} - Q_i).$$

This is the classical first-order centred form. In the present study it serves as the most diffusive baseline among the three methods.

The Lax-Wendroff method is implemented in its two-step Richtmyer form. First, an interface state is predicted through

$$Q_{i+1/2}^{n+1/2} = \frac{1}{2} (Q_i^n + Q_{i+1}^n) - \frac{\Delta t}{2\Delta x} [F(Q_{i+1}^n) - F(Q_i^n)],$$

The numerical flux is then obtained by evaluating the physical flux at that predicted state,

$$\hat{F}_{i+1/2}^{LW} = F(Q_{i+1/2}^{n+1/2}).$$

For smooth solutions, this scheme is second-order accurate. However, in its unrestricted form it is not monotone, and oscillatory overshoots can occur near discontinuities.

The FORCE flux is defined as the arithmetic average of the Lax-Friedrichs and Lax-Wendroff-type interface fluxes,

$$\hat{F}_{i+1/2}^{FORCE} = \frac{1}{2} (\hat{F}_{i+1/2}^{LF} + \hat{F}_{i+1/2}^{LW}).$$

The implementation considered here is first order. No general claim is made that it is TVD, monotone, or free from oscillations, since such properties depend on the complete numerical formulation and the assumptions under which it is analyzed.

4.3 Time stepping, boundaries, and admissibility

The acoustic speed is evaluated in each cell as

$$c_i = \sqrt{\frac{k\gamma(m_i + n_i)^\gamma}{n_i}},$$

The maximum characteristic-speed estimate used in the CFL condition is

$$a_{\max} = \max_i (|u_i| + c_i).$$

The time step is then chosen as

$$\Delta t = \min \left(C_{\text{CFL}} \frac{\Delta x}{a_{\max}}, T - t \right).$$

Basically, baseline calculations use $C_{\text{CFL}} = 0.5$. The absolute-value form of the spectral bound is required because a test or parameter choice can

contain a left-moving first-family wave even when u is positive. The same Δt used in the state update advances the numerical clock, and the final step is truncated to terminate exactly at the requested final time.

For the Riemann problems, zero-gradient ghost cells are used at the boundaries, with the exterior state copied from the nearest interior cell. The smooth translating-contact problem uses periodic boundary conditions. Spatial arrays are handled with explicit indexing along the space direction to avoid mixing conserved components through an axis-unspecified cyclic array operation.

The phase masses are checked throughout the computation. Every accepted state must satisfy $m > 0$ and $n > 0$, and intermediate states required by a flux calculation are checked where appropriate. If either phase mass becomes non-positive, the run is stopped and recorded as a failure. No artificial density floor or clipping is used.

4.4 Model-specific Riemann reference

For each initial side $s \in \{L, R\}$, define

$$r_s = \frac{m_s}{n_s}, \quad p_s = k(m_s + n_s)^\gamma, \quad A_s = k(1 + r_s)^\gamma.$$

Because the ratio m/n is constant along an acoustic wave,

$$p = A_s n^\gamma,$$

For a trial pressure p , the corresponding continuation of the liquid-phase mass from side s is

$$\hat{n}_s(p) = n_s \left(\frac{p}{p_s} \right)^{1/\gamma}.$$

The change in velocity across the wave is represented by the scalar function

$$f_s(p) = \begin{cases} \sqrt{(p - p_s) \left(\frac{1}{n_s} - \frac{1}{\hat{n}_s(p)} \right)}, & p > p_s, \\ \frac{2c_s}{\gamma - 1} \left[\left(\frac{p}{p_s} \right)^{(\gamma-1)/(2\gamma)} - 1 \right], & 0 < p \leq p_s, \gamma \neq 1, \\ c_s \ln(p/p_s), & 0 < p \leq p_s, \gamma = 1. \end{cases}$$

The star pressure p^* is obtained from the scalar equation

$$f_L(p_*) + f_R(p_*) + u_R - u_L = 0,$$

Once p^* is known, the velocity in the star region is

$$u_* = \frac{u_L + u_R + f_R(p_*) - f_L(p_*)}{2}.$$

The numerical root is bracketed on the positive-pressure interval. After the root is found, the two star states are constructed and each outer wave is classified as a shock or rarefaction according to whether p^* is greater or smaller than the corresponding initial pressure. When a rarefaction is present, the complete fan is sampled. Because the finite-volume schemes evolve cell averages, the reference solution is also converted into cell averages before errors are calculated. Eight-point Gaussian quadrature is used, with integration intervals split at shocks, fan edges, and the contact discontinuity whenever these features pass through a cell.

4.5 Error, convergence, and conservation diagnostics

For a conserved or diagnostic variable z , the reported errors are

$$E_1(z) = \Delta x \sum_i |z_i - \bar{z}_i^{ref}|,$$

$$E_2(z) = \left(\Delta x \sum_i |z_i - \bar{z}_i^{ref}|^2 \right)^{1/2},$$

and

$$E_\infty(z) = \max_i |z_i - \bar{z}_i^{ref}|.$$

Velocity is recovered from each numerical state as $u = j/n$. The observed order under grid doubling is

$$p_{obs} = \frac{\log(E_N/E_{2N})}{\log 2}.$$

The discrete conservation diagnostic includes boundary fluxes. If $M^n = \Delta x \sum_i Q_i^n$, the residual is

$$R_{cons} = M^{N_t} - M^0 + \sum_k \Delta t^k (\hat{F}_{right}^k - \hat{F}_{left}^k).$$

The largest component of this residual is reported. Including the boundary fluxes is necessary because the Riemann calculations use outflow boundaries. The change of the conserved quantity inside the domain must balance the accumulated numerical flux through its endpoints.

5. Analysis and Discussion

5.1 Baseline Riemann problems and wave structure

The first principal test uses

$$W_L = (1.5, 1.8, 2.0)^T, \quad W_R = (1.2, 1.0, 2.5)^T, \quad T = 0.15.$$

The second uses,

$$W_L = (0.8, 1.0, 1.2)^T, \quad W_R = (1.0, 0.8, 1.0)^T, \quad T = 0.25.$$

For the baseline computations, $k = 0.6$, $\gamma = 0.8$, and CFL = 0.5. The model-specific Riemann construction gives the wave structures summarized in Table 1.

Table 1. Riemann wave structure and star states for the baseline problems ($k = 0.6$, $\gamma = 0.8$).

Problem	First wave	Middle wave	Third wave	p_*	u_*
1	Rarefaction	Contact	Rarefaction	1.053113	2.418704
2	Shock	Contact	Shock	1.047103	1.105573

For Test Problem 1, the left rarefaction fan spans speeds 1.167489-1.544323, the contact speed is 2.418704, and the right fan spans 3.376541-3.449707. At $T = 0.15$, the first fan is located approximately between $x = 0.1751$ and $x =$

0.2316, the contact is near $x = 0.3628$, and the right fan lies between $x = 0.5065$ and $x = 0.5175$. The correct wave pattern is therefore rarefaction-contact-rarefaction (R-C-R).

For Test Problem 2, the initial total phase mass is 1.8 on both sides and the initial pressures are equal, even though phase composition and velocity differ. The star pressure is 1.047103, which exceeds both initial pressures, so both acoustic waves are compressive shocks. The shock

and contact speeds are 0.279863, 1.105573, and 2.028745. At $T = 0.25$, the corresponding positions are approximately $x = 0.06997$, 0.27639, and 0.50719. The resulting pattern is shock-contact-shock (S-C-S).

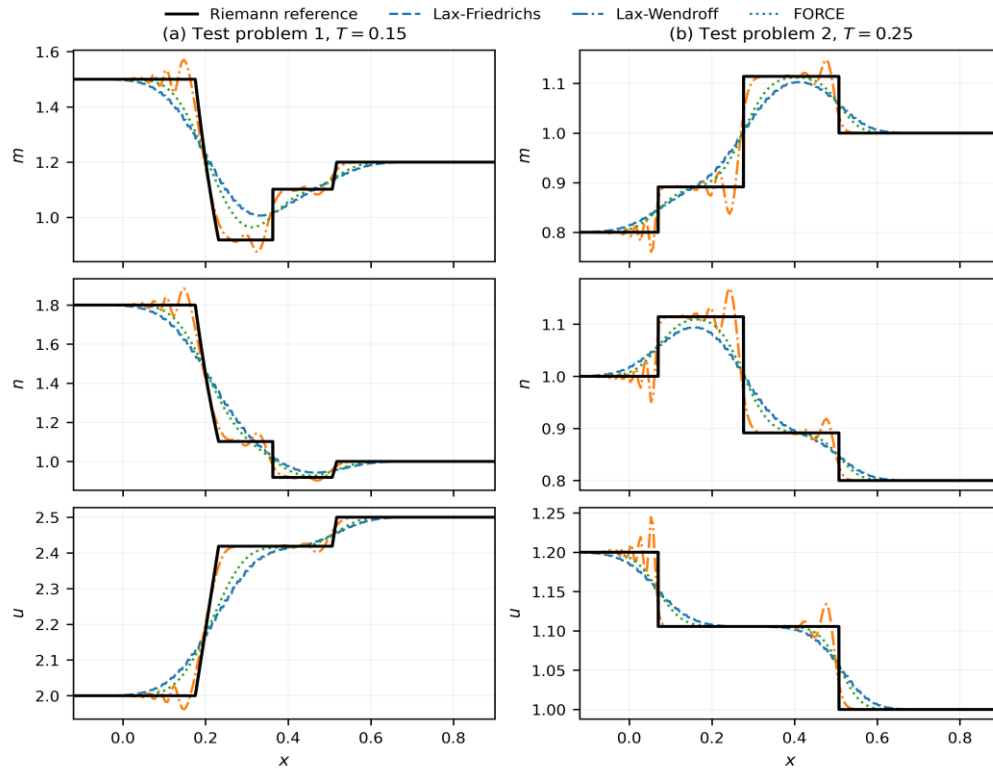


Figure 1. Model-specific Riemann reference and numerical solutions for the two baseline problems at $N = 400$, $k = 0.6$, $\gamma = 0.8$, and $CFL = 0.5$. The left column corresponds to Test Problem 1 at $T = 0.15$, and the right column to Test Problem 2 at $T = 0.25$. Gas-phase mass m , liquid-phase mass n , and velocity u are shown.

The numerical profiles make the differences among the three schemes easy to see. Lax-Friedrichs spreads sharp features over several cells, especially at the contact and at the two shocks in the second problem. Lax-Wendroff gives a much sharper representation but develops small oscillatory excursions close to discontinuities. FORCE produces a result between these two extremes. This comparison

also shows why a wave should not be classified from the apparent smoothness of a numerical curve: a shock broadened by numerical diffusion remains an approximation of a shock.

5.2 Quantitative comparison at $N = 400$

The integral L^1 errors for the two principal test problems are listed in Table 2.

Table 2. L^1 errors at $N = 400$ for the two baseline Riemann problems.

Problem	Method	$E_1(m)$	$E_1(n)$	$E_1(u)$
1	Lax-Friedrichs	4.216×10^{-2}	4.174×10^{-2}	2.615×10^{-2}
1	Lax-Wendroff	1.146×10^{-2}	1.255×10^{-2}	5.755×10^{-3}

Problem	Method	$E_1(m)$	$E_1(n)$	$E_1(u)$
1	FORCE	2.998×10^{-2}	3.103×10^{-2}	1.805×10^{-2}
2	Lax-Friedrichs	2.094×10^{-2}	2.161×10^{-2}	9.817×10^{-3}
2	Lax-Wendroff	7.511×10^{-3}	7.504×10^{-3}	3.526×10^{-3}
2	FORCE	1.507×10^{-2}	1.531×10^{-2}	6.680×10^{-3}

For all three reported variables, the same ordering appears in both problems: $E_{LW} < E_{FORCE} < E_{LF}$. In Test Problem 1, the Lax-Wendroff gas-mass error is about 72.8% smaller than the Lax-Friedrichs error and about 61.8% smaller than the FORCE error. In Test Problem 2, the corresponding reductions are approximately 64.1% and 50.2%.

These values give Lax-Wendroff the smallest global error in the two baseline experiments, but the error table should be read together with the solution profiles. Lax-Wendroff resolves the wave transitions more sharply, yet the same low numerical dissipation allows small overshoots and undershoots around discontinuities. Lax-Friedrichs suppresses most visible oscillations by adding stronger numerical viscosity, but this spreads shocks and contacts over a wider region and increases the integral error.

Contact discontinuity deserves separate attention. Across the exact contact, velocity and pressure remain continuous, whereas m and n may jump. An apparently accurate velocity curve can therefore hide a poorly resolved phase-composition contact. In the present calculations,

Lax-Friedrichs smears this contact most strongly, FORCE improves its resolution, and Lax-Wendroff produces the sharpest representation among the three methods.

5.3 Mesh refinement and smooth-contact verification

The two Riemann problems were also solved on $N = 100, 200, 400$, and 800 cells. For all three methods, the gas-mass L1 error decreases as the grid is refined. Since the exact Riemann solutions contain shocks and contacts, however, these experiments are not used to measure the formal smooth-solution order of the schemes.

To obtain a cleaner convergence check, a separate smooth translating-contact solution is used on the periodic interval $[-1, 1]$:

$$m(x, t) = 1 + 0.2 \sin(\pi(x - ut)), \quad n(x, t) = 1 - 0.2 \sin(\pi(x - ut)),$$

with $u = 0.7$ and $T = 0.4$. In this test $m + n = 2$ for all x and t , so the pressure is constant and the phase composition is transported by the linearly degenerate field. Exact cell averages are used at both the initial and final times.

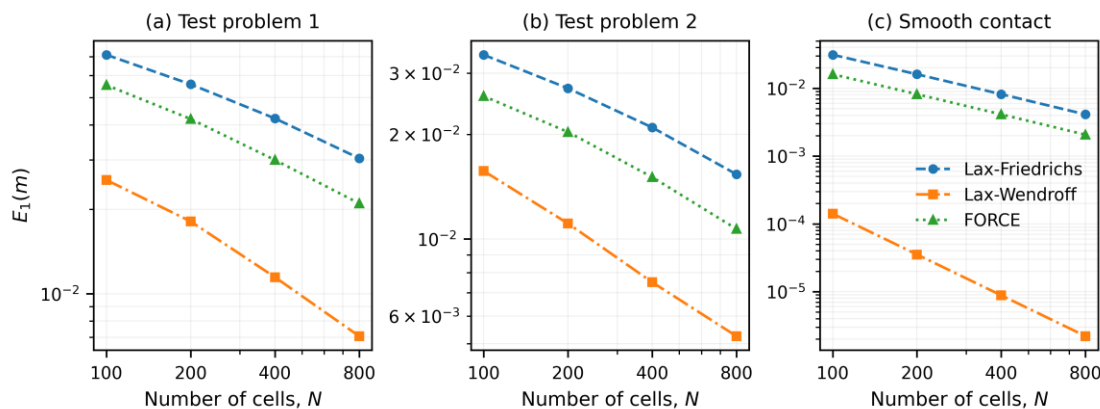


Figure 2. Grid-refinement results for the gas-phase mass error $E_1(m)$ on $N = 100, 200, 400$, and 800 cells. The panels compare Test Problem 1, Test Problem 2, and the smooth translating-contact problem.

At $N = 800$, the smooth-contact results are summarized in Table 3.

Table 3. Smooth-contact $L_1(m)$ error at $N = 800$ and observed order from $N = 400$ to 800.

Method	$E_1(m)$	Observed order 400 $\rightarrow$ 800
Lax-Friedrichs	4.115×10^{-3}	0.991
Lax-Wendroff	2.208×10^{-6}	2.000
FORCE	2.066×10^{-3}	0.997

The measured values are very close to first order for Lax-Friedrichs and FORCE and second order for Lax-Wendroff. At the final refinement step, doubling the number of cells therefore reduces the LF and FORCE errors by approximately a factor of two, while the LW error falls by approximately a factor of four.

The smooth test also helps explain the behavior seen in the Riemann problems. With no shock present, the main error in the two first-order schemes is gradual damping and distortion of the transported wave. Lax-Wendroff preserves the smooth profile much more accurately. In the discontinuous cases, however, its lower dissipation allows oscillations to appear close to shocks and contacts. These observations describe two different aspects of the same scheme rather than a contradiction.

The convergence result should nevertheless be kept within its proper scope. The test mainly excites the linearly degenerate contact mode. It

confirms the expected implementation order for this smooth solution but does not by itself establish the same empirical order for every nonlinear acoustic wave of the full system.

5.4 Sensitivity to the pressure exponent

The pressure exponent influences the acoustic speed through

$$c^2 = \frac{k\gamma(m+n)^\gamma}{n}.$$

The pressure exponent changes the acoustic speed and therefore affects the characteristic wave curves as well as the time-step restriction. To examine this sensitivity, the two principal problems were recomputed with $\gamma = 1.4$ while retaining $k = 0.6$ and the same left and right states. This is a mathematical sensitivity test only; $\gamma = 1.4$ is not presented as a calibrated physical parameter. The resulting Riemann structures are summarized in Table 4.

Table 4. Effect of the pressure exponent γ on the Riemann wave structure and star states.

Problem	γ	First wave	Third wave	p_*	u_*
1	0.8	Rarefaction	Rarefaction	1.053113	2.418704
1	1.4	Rarefaction	Shock	1.906145	2.559420
2	0.8	Shock	Shock	1.047103	1.105573
2	1.4	Shock	Shock	1.502288	1.105573

The change is especially noticeable in Test Problem 1. With $\gamma = 0.8$, the wave pattern is R-C-R, whereas at $\gamma = 1.4$ it becomes R-C-S; the right rarefaction has changed into a shock. For Test Problem 2, the wave types remain S-C-S, but the

left shock changes direction. Its speed is 0.279863 for $\gamma = 0.8$ and approximately -0.240576 for $\gamma = 1.4$, while the right-shock speed at $\gamma = 1.4$ is approximately 2.610613.

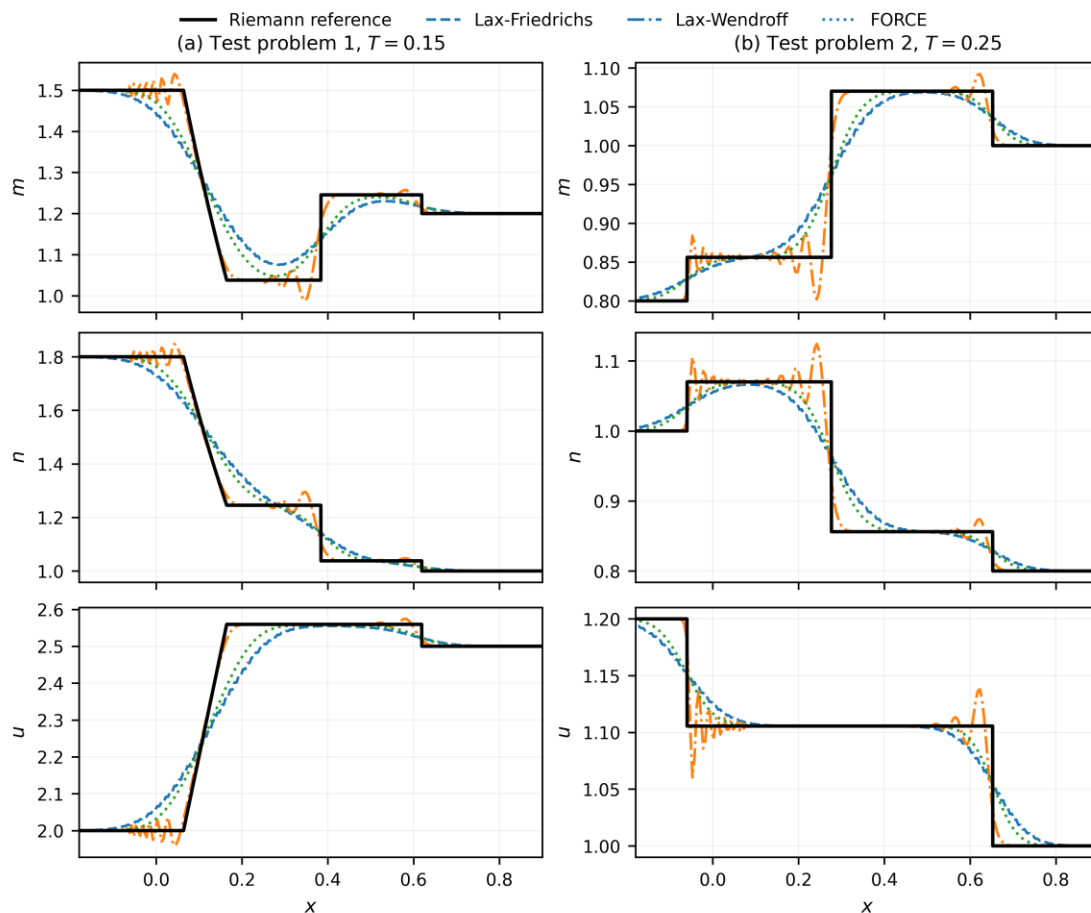


Figure 3. Parameter-sensitivity results for $\gamma = 1.4$, $N = 400$, $k = 0.6$, and $CFL = 0.5$. Test Problem 1 changes to a rarefaction-contact-shock structure, whereas Test Problem 2 remains shock-contact-shock and contains a left-moving first shock.

The left-moving shock is also a useful numerical check. It shows why the CFL estimate should account for the magnitude of negative characteristic speeds. The characteristic-speed bound used in Section 4.3 handles both directions without assuming that the fastest relevant wave travels to the right.

The sensitivity study also makes clear that results obtained with different values of gamma cannot simply be relabeled. The earlier computations used $\gamma = 0.8$, whereas later text in the preprint referred to a regime with $\gamma > 1$ (Rab, 2025). Since changing the exponent changes the Riemann solution itself, the present study retains $\gamma = 0.8$ as the baseline and reports the $\gamma = 1.4$ calculations separately.

5.5 Robustness and independent verification

The numerical study contains more than the two principal $N = 400$ calculations. The main benchmark set consists of 48 Riemann refinement runs, 12 smooth-contact convergence runs, and 12 additional calculations. The supplementary cases include left-moving waves, a stationary contact, a mixed shock-rarefaction configuration, and positive low-density data. All 72 main benchmark runs finished without producing a non-positive accepted phase mass. No density clipping or fallback flux was used.

This is useful evidence that the implementation behaves robustly on the selected test set, but it should not be interpreted as a general positivity theorem. A finite collection of successful calculations cannot establish preservation of $m >$

0 and $n > 0$ for arbitrary initial data, particularly The analytical reference and numerical implementation were also checked

close to a vacuum limit. independently. The largest recorded discrepancies are listed in Table 5.

Table 5. Independent verification diagnostics and largest recorded discrepancies.

Verification quantity	Largest discrepancy
Conservative-flux eigenvalue check	6.259×10^{-10}
Characteristic directional derivative	4.177×10^{-10}
Shock Rankine-Hugoniot residual	1.198×10^{-15}
Contact jump residual	1.110×10^{-15}
Rarefaction PDE residual	4.352×10^{-9}
Star-pressure root residual	6.140×10^{-14}
8-point vs. 16-point reference quadrature	1.332×10^{-15}
Boundary-flux conservation residual	2.276×10^{-15}

The flux Jacobian was differentiated numerically and its eigenvalues were compared with $u - c$, u , and $u + c$. The directional derivatives used to classify the characteristic fields were checked separately. Shock states were substituted directly into the three Rankine-Hugoniot conditions, and the corresponding contact relations were also verified. Inside rarefaction fans, sampled states were checked against the corresponding self-similar differential relation. These tests examine different parts of the implementation rather than relying only on agreement between the numerical solution and a reference generated by the same code.

The pressure-root residual is much smaller than the finite-volume errors reported earlier. Similarly, repeating selected reference integrations with 16-point quadrature instead of the standard eight-point rule changes the cell averages only at approximately machine-precision scale. The reported error measurements are therefore not controlled by root-solving or quadrature error.

The conservation diagnostic gives the largest residual of 2.276×10^{-15} . This is at the level expected from floating-point arithmetic and confirms the conservative balance, including the numerical boundary fluxes, for the benchmark

calculations. Uniform-state and reflected tests also agree with their expected behavior to machine precision.

5.6 Overall comparison and limitations

Looking at the experiments, the three schemes show a fairly consistent pattern on the two baseline problems. Lax-Friedrichs introduces the largest amount of numerical diffusion. Its shocks and contacts are spread over wider regions of the grid, and this is reflected in the larger integral errors. The same dissipation, however, strongly damps oscillatory numerical structure.

Lax-Wendroff behaves almost oppositely. It gives the smallest measured L1 errors at $N = 400$ and reproduces second-order convergence on the smooth contact test. Its discontinuities are sharper, but localized overshoots and undershoots are visible. This is why the error norms and solution profiles should be considered together rather than using either one by itself.

FORCE gives an intermediate result in the present calculations. It is less diffusive than Lax-Friedrichs and gives smaller errors, but its wave profiles remain more diffuse than those of Lax-Wendroff. No general statement of TVD or monotonicity is made for the particular FORCE implementation used here.

The comparison is therefore better viewed as a trade-off among accuracy, diffusion, dispersion, and wave resolution. No single error measure describes all four. This is particularly relevant for the present two-phase model because the contact discontinuity changes the individual phase masses while leaving pressure and velocity continuous. A scheme that looks satisfactory in the velocity profile may still represent the phase composition poorly.

The conclusions also have several limitations. The physical calibration of the reduced pressure closure has not been independently established here. The main comparison includes only three classical schemes, while modern limited and higher-order methods may behave differently. The low-density experiments are not a systematic investigation of the near-vacuum limit, and no positivity, entropy-stability, or nonlinear convergence theorem is proved. The smooth convergence test concerns one exact contact mode rather than all characteristic families, and computational efficiency is not ranked because isolated wall-clock timings would not provide a fair comparison without repeated measurements at comparable error levels.

Within these limits, the corrected benchmark provides a useful starting point for further work. A higher-resolution method can be tested against the same Riemann reference, error definitions, conservation balance, and parameter-sensitive wave classification, making any future improvement easier to assess on a consistent basis.

6. Conclusion

This study revisited the numerical solution of a reduced liquid-gas two-phase isentropic flow model using three classical finite-volume schemes: Lax-Friedrichs, two-step Lax-Wendroff, and FORCE. Particular attention was given to correcting the mathematical interpretation of the characteristic structure and to making the numerical implementation consistent across all three methods. The characteristic analysis gives the wave speeds $u - c$, u , and $u + c$, with the first and third fields genuinely nonlinear and the middle field linearly degenerate. This distinction

determines which waves may appear as shocks or rarefactions and which one represents the contact discontinuity.

For the baseline choice $k = 0.6$ and $\gamma = 0.8$, Test Problem 1 has a rarefaction-contact-rarefaction structure and Test Problem 2 contains two shocks separated by a contact. At $N = 400$, Lax-Wendroff gives the smallest measured integral errors for m , n , and u in both cases. Lax-Friedrichs is more diffusive, whereas FORCE gives intermediate errors and wave resolution. The sharper Lax-Wendroff profiles are accompanied by visible oscillations near discontinuities, so the smallest global error does not automatically imply the best local behavior.

The smooth translating-contact test provides a separate check of the implemented convergence behavior. Between $N = 400$ and $N = 800$, the observed orders are approximately 0.991, 2.000, and 0.997 for Lax-Friedrichs, Lax-Wendroff, and FORCE, respectively. These values are consistent with first-order behavior for Lax-Friedrichs and FORCE and second-order behavior for Lax-Wendroff for the smooth mode considered here.

The sensitivity study also shows that the pressure exponent γ has a direct effect on the Riemann solution. Increasing gamma from 0.8 to 1.4 changes the right wave in Test Problem 1 from a rarefaction to a shock, while the first shock in Test Problem 2 changes its direction of propagation. Calculations performed with different values of gamma therefore cannot be interpreted as if they belonged to the same wave configuration.

The broader benchmark set provides additional confidence in the corrected implementation. Across 72 main runs, no accepted state developed a non-positive phase mass, and the largest conservation residual was 2.276×10^{-15} . The shock, contact, rarefaction, eigenvalue, pressure-root, and quadrature checks also remained small. These results support the numerical and reference implementations for the test cases considered, although they should not be read as general proofs of positivity or convergence.

Taken together, the results show that the three methods offer different compromises. Lax-Friedrichs is robust but strongly diffusive. Lax-

Wendroff is more accurate on the present tests and performs especially well for smooth contact, but its low dissipation allows oscillations near discontinuities. FORCE lies between the two in both error and resolution. A natural next step is to examine limited higher-resolution methods that reduce contact and shock smearing while controlling oscillations. Further work should also consider positivity-preserving or invariant-domain strategies, smooth nonlinear acoustic waves, stronger low-density and near-vacuum configurations, and a clearer physical interpretation and parameter range for the pressure closure.

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