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DISCRETE-TIME SIGNAL ANALYSIS USING THE Z-TRANSFORM AND INVERSE Z-TRANSFORM VIA THE CAUCHY RESIDUE THEOREM: APPLICATIONS TO DIFFERENCE EQUATIONS AND RECURSIVE SEQUENCES

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Abstract

Z-transform has been used as an analysis tool for years. Our motive is to deeply study this technique and all its basics to solve any type of discrete-time signal. In addition, we are focused to solve the discrete domain difference equations.Z-transform has a specialty that is it can modify itself according to the nature of signal or system. First, we have studied basic elements and the crucial aspects of inspecting a system. The important element of mapping a real domain function into a complex domain is to view it in a region of convergence. This region defines that the function is analytic in it. That is how we can create an outline of that transformed function. The introduction, significance and applications of Z-transform are briefly discussed in Chapter 1. While we further studied the properties of Z-transform along with their proofs. The inverse Z-transform is the main step in order to solve difference equations. Although there are different techniques of calculating the inverse Z-transform, but we have applied the methodology of Cauchy residue theorem. The solution of famous difference equation is detailed in Chapter 3. We have concluded that for discrete domain Z-transform is best for analysis and inverse Z-transform is advantageous for developing the compact forms of many recursive relations like Legendre polynomial, Fibonacci sequence and Chebyshev difference equation.

Chapter 1
Introduction

Z-transform is a popular tool used to analyze any discrete time linear dynamical system. The analysis of discrete signal is crucial for system design and its stability. The Z-transform has basic idea from Laplace transform. Z-transform was first inaugurated by W. Hurewicz in 1947[1]. In 1952, Ragazzini and Zadeh proposed the name Z-transform, in a group of sampled-data control at University of Columbia [2]. Later, E.I. Jury developed Advanced Z-transform [3]. Z-transform techniques have now become an important tool in Electronics, Computer and Communicating engineering due to rapid development of Integral circuit technology and microprocessor architecture. According to mathematical literature, idea of Z-transform is contained in method of generating functions. Z-transform is a generalized form of Fourier transform and it is an outlook of Laurent series in variable z. The benefit of this transformation is that it has ability to modify according to the nature of signal either continuous or discrete.

Most applications of Z-transform are in processing digital signals, population sciences and control theory. Usually it is used to analyse and process digital data. In field of digital signal processing it is used in designing system and digital filters, further to check frequency feedback and system's stability. In telecommunication automatic controls use this technique in order to intensify the electrical and mechanical energy which results dynamic nature of the system. By the help of this tool one can solve the behaviour equation of any control system furthermore one can investigate the feedback of the system. The simulation of continuous control system is also done by using Z-transform, one can analyse the system from different aspects. In sampled-time control plants it deals with the invariant-difference equations.

Chapter 2
Preliminaries and Definitions
2.1 Z-transform

The Z-transform is a method of converting a time depended discrete signal x[n] into the form of complex function, i.e.,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}, \quad z \in D \qquad (2.1)$$

where D shows domain of complex variable z∈C, such that the series converges in it. The priary task of Z-transform is mapping a complex frequency domain representation from a discrete time signal. If $z = re^{i\theta}$, then X(z) evaluated at r=1 becomes the Fourier Transform,

$$F(z) = \sum_{n=-\infty}^{\infty} x[n] e^{-ni\theta}$$

So, we conclude that a discrete-time Fourier transform is a special case of Z-transform. Z-transform is also a discrete analogue of the Laplace transform.

2.2 Region of convergence

Region of convergence (ROC) is a range where Z-transform is analytic. On all points of ROC, the existence of Z-transform is mandatory.

2.2.1 Casual ROC

A region of convergence in which the origin is punched out is called the Casual Region of Convergence. That is all over *z*-plane except at origin.

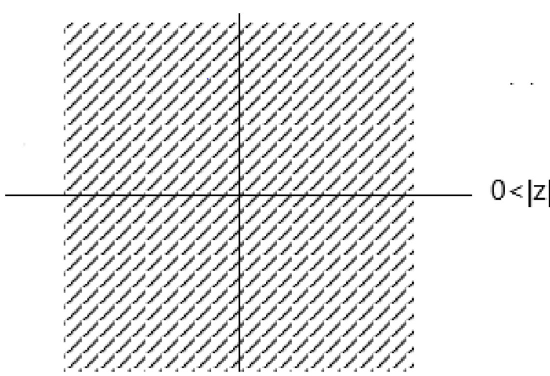

Fig 2.1: ROC whole Z-plane except origin

2.2.2 Anti-casual ROC

The region of convergence that is contained in unit circle is called anti-casual region of convergence.

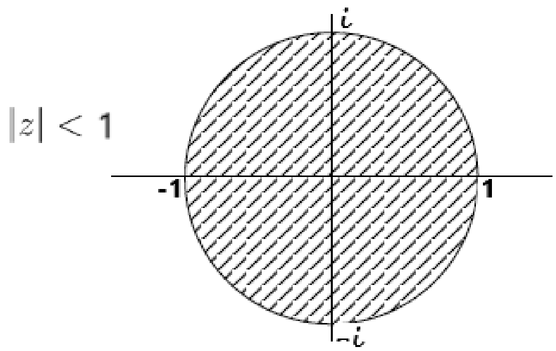

Fig 2.2: ROC contained in a unit circle

2.3 Properties of Region of Convergence

1. If ROC of the Z-transform of any discrete-time signal contains a circle of radius 1, then discrete- time Fourier transform also exists.
2. Poles are not included in ROC.
3. ROC of a finite sequence is whole z-plane excluding the points z=0, ∞.
4. If x[n] is positive sequence, then ROC is $|z| > |\ell|$; where ℓ is the pole of largest magnitude in X(z).
5. If x[n] is left-sided signal, then ROC is given by $|z| > |u|$; where u is the pole of smallest magnitude in X(z).
6. If x[n] is double-sided signal, then ROC is given by $|\ell| < |z| < |u|$; where ℓ and u are poles in E(z), i.e., $|\ell| < |u|$.
7. The ROC is always a connected region.
8. Some common shapes of ROC are:
   i. Whole Z-plane
   ii. Ring
   iii. Inside a circle (see Fig 2.2)
   iv. Outside a circle
   v. Only at center of Z-plane, i.e., z=0

2.4 Inverse Z-transform

Consider a transformed function X(z) and its region of convergence, then we can find its signal or sequence x[n] by taking inverse Z-transform, defined as

$$x[n] = Z^{-1}[X(z)] = \frac{1}{2\pi i}\oint_c$$

where C is an anti-clockwise closed path that encircles origin. C also exists inside ROC. If ROC is causal, then all the poles must lie inside path C. If path C is a unit circle exactly, then it is special case of contour integral. The contour is used when region of convergence contains the unit circle, that is assured if system is stable, or simply the unit circle includes all the poles. For given ROC, Z-transform and inverse Z-transform are one-to-one mappings. Observe that discrete analogue of inverse Laplace transform is inverse Z-transform.

2.5 Theorem

If X(z) is Z-transform of a sequence $\{X_k = x[k]\}\Big|_{k=0}^{\infty}$ defined in the region $R < |z|$. Then $x[n]$ is given by the formula,

$$X_k = x[k] = \frac{1}{2\pi i}\oint_c z^{k-1} X(z)dz \,. \qquad (2.3)$$

Here C is any positive oriented curve that is closed and lies in region $R < |z|$ and circulates around the origin.

2.6 Cauchy's Residue Theorem

If X(z) is a transformed function of x[n] along with ROC R_x, then the inverse transform is defined by

$$x[n] = \frac{1}{2\pi i}\oint_c X(z)dz = \sum Residue[X(z), z_i] \,, \qquad (2.4)$$

where the integration is performed in anticlockwise path around z=0 in ROC of X(z).

2.6.1 Corollary

If Z-transform of $x[n]$ is represented by X(z), then this signal is determined by the formula,

$$x[n]=\sum_{k=1}^{j}\operatorname{Res}\left(X(z)z^{n-1},z_k\right) \tag{2.5}$$

Here $f(z)=X(z)z^{n-1}$ has poles at $z_k,\ k=1,2,\ldots,j$.

2.6.2 Corollary

Let Z-transform of sequence x[n] is represented by X(z). If X(z) has simple poles at $z_1,z_2,\ldots,z_j$ then $x[n]$ is given by the formula

$$x[n]=\sum_{k=1}^{j}\left[\lim_{z\to z_k}\left(z-z_k\right)X(z)z^{n-1}\right]$$

2.6.3 Corollary

Let Z-transform of sequence x[n] is represented by X(z) and z_0 be pole whose order is n then $x[n]$ is determined by

$$x[n]=\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n X(z)z^{n-1} \tag{2.6}$$

2.7 Analysis of a system

For inspecting any discrete-time dynamical system it is necessary to analyse its region of convergence. Following are the aspects of inspecting such system.

2.7.1 Stability of a system

If unit circle is contained in ROC, then system is said to be stable.

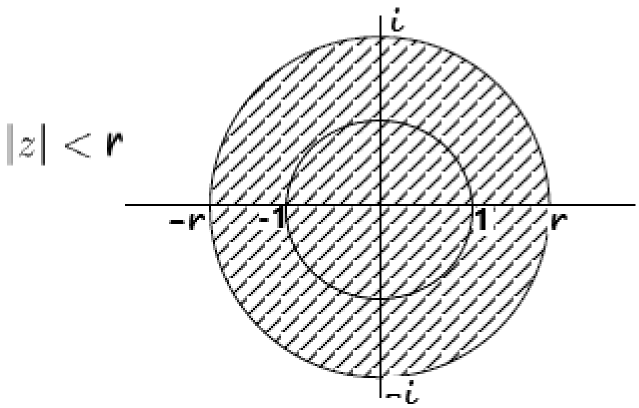

Fig 2.3: ROC that contains the unit circle

2.7.2 Causality of a system

The system is causal if ROC contains infinity as well as the signal is right-sided sequence.

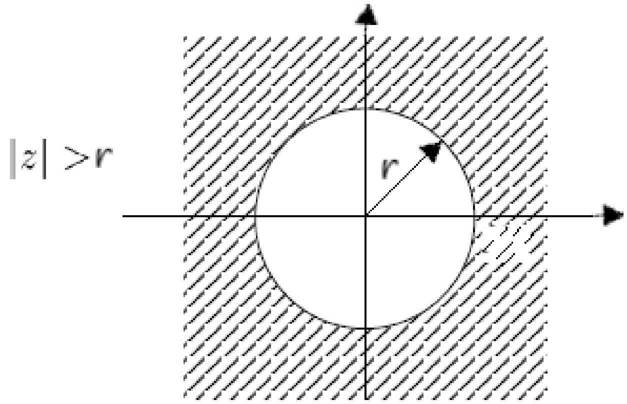

Fig 2.4: causal system

The system is anti-causal if ROC contains origin and the signal is left-sided sequence.

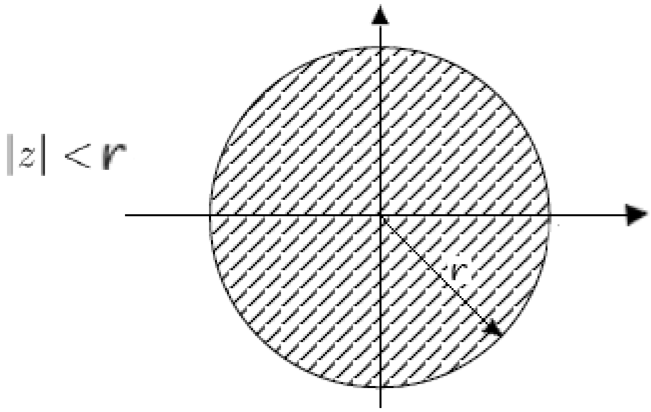

Fig 2.5: anti-causal system

2.8 Solved example of inverse Z-transform

Using the Cauchy's Residue Theorem ,consider transformed function X(z) of the form

$$X(z)=\frac{1-\frac{1}{z}}{\left(-\frac{1}{3}z^{-1}+1\right)\left(-\frac{1}{2}z^{-1}+1\right)}$$

Simplify using the partial fractions method, we have

$$X(z)=\frac{4}{\left(-\frac{1}{3}z^{-1}+1\right)}-\frac{3}{\left(-\frac{1}{2}z^{-1}+1\right)}$$

Apply inverse Z-transform, after that use the Cauchy's Residue Theorem 2.6

$$x[n]=\frac{1}{2\pi i}\oint_c\left(\frac{-3}{\left(1-\frac{1}{2}z^{-1}\right)}+\frac{4}{\left(1-\frac{1}{3}z^{-1}\right)}\right)z^{n-1}dz,$$

we get two poles here as $z=\frac{1}{2}$ and $z=\frac{1}{3}$. There are three possibilities of ROC. The cases are given below.

Case (i) ROC: $|z| > \frac{1}{2}$

By using (2.5) and (2.6), we have

$$x[n]=Residue\left(X(z)z^{n-1},\frac{1}{2}\right)u[n]+Residue\left(X(z)z^{n-1},\frac{1}{3}\right)u[n], \qquad (2.7)$$

where

$$\operatorname{Res}\left(X(z)z^{n-1},\frac{1}{2}\right)=\lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\left(\frac{(1-z^{-1})z^{n-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}\right)=-3\left(\frac{1}{2}\right)^{n}$$

and

$$\operatorname{Res}\left(X(z)z^{n-1},\frac{1}{3}\right)=\lim_{z\to\frac{1}{3}}\left(-\frac{1}{3}+z\right)\left(\frac{(1-\frac{1}{z})z^{n-1}}{\left(-\frac{1}{2}z^{-1}+1\right)\left(-\frac{1}{3}z^{-1}+1\right)}\right)=4(3)^{-n}$$

So (2.7) becomes

$$x[n]=-3(2)^{-n}u[n]+4(3)^{-n}u[n]$$

Case (ii) ROC: $\frac{1}{3}<|z|<\frac{1}{2}$

We get same poles $\frac{1}{2},\frac{1}{3}$ as in previous case (i) and (ii). By the theorem 2.6

$$x[n]=Residue\left(X(z)z^{n-1},\frac{1}{2}\right)u[-1-n]+Residue\left(X(z)z^{n-1},\frac{1}{3}\right)u[n]$$

$$x[n]=3(2)^{-n}u[-1-n]+4(3)^{-n}u[n]$$

Case (iii) ROC: $|z|<\frac{1}{3}<\frac{1}{2}$

We get the same poles $\frac{1}{2},\frac{1}{3}$ and same values of residue evaluation on each pole.

$$x[n]=3(2)^{-n}u[-1-n]-4(3)^{-n}u[-1-n]$$

2.9 Properties of Z-transform

2.9.1. Property of Linearity

$$am[t]+bn[t]$$

Apply Z-transform as

$$Z(a\,m[t]+b\,n[t])=\sum_{t=0}^{\infty}z^{-t}\left(a\,m[t]+b\,n[t]\right)$$

$$=a\sum_{t=0}^{\infty}z^{-t}\,m[t]+b\sum_{t=0}^{\infty}z^{-t}\,n[t]=aZ(m[t])+bZ(n[t])$$

2.9.2 Unit-Step Function

Unit-step function in discrete domain is stated by

$$u[n]=\begin{cases}0, & n<c\\ 1, & n>c\end{cases} \tag{2.8}$$

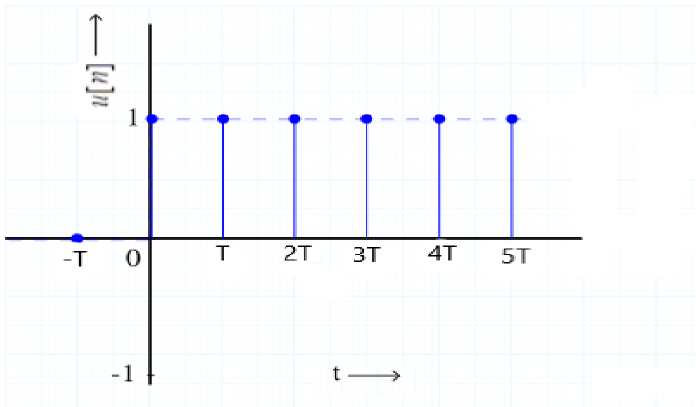

Fig 2.6: unit step function

Now, applying *z*-transform on (2.8) as;

$$Z\left[u[n]\right]=\sum_{n=0}^{\infty}\left(u[n]z^{-n}\right)$$

as $u[n]=1$ as defined in (2.8), and

$$Z(u[n])=\sum_{n=0}^{\infty}z^{-n}=\sum_{n=0}^{\infty}(z^{-1})^{n}$$

Let $t=\dfrac{1}{z}$ and by expansion of power series in terms of 't', we have;

$$\sum_{n=0}^{\infty}t^{n}=\frac{1}{1-t},\ |t|<1$$

$$=\frac{1}{\left(1-\frac{1}{z}\right)},\ \left|\frac{1}{z}\right|<1$$

Simplify

$$Z\left[u[n]\right]=\frac{z}{(-1+z)},|z|>1$$

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2.9.3 Time Delay

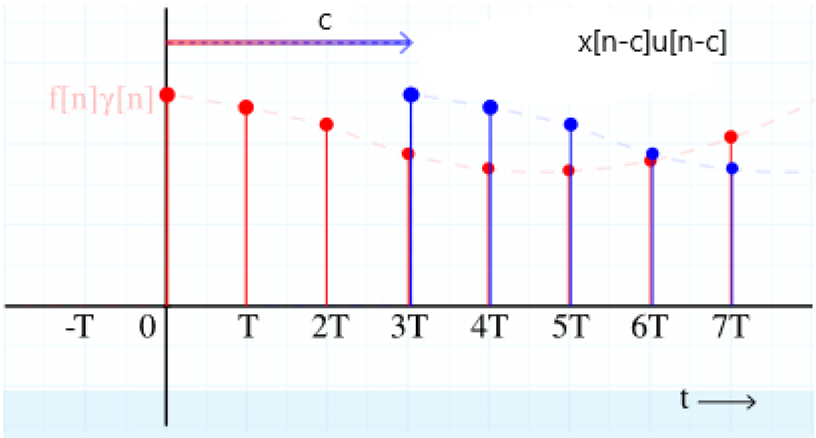

Fig 2.7: Time delayed by 'c' units

Consider $x[n-c]u[n-c]$,that is delayed 'c' units, c >0. Also, we truncate the sequence at n=0.
Here $u[n-a]$ denotes unit-step function that describes delay and is stated as;

$$u[n-c]=\begin{cases}0, & n<c\\ 1, & n>c\end{cases} \tag{2.9}$$

Now, applying Z-transform

$$Z\,[x[n-c]u[n-c]]=\sum_{n=0}^{\infty} z^{-n}\left(x[n-c]u[n-c]\right)$$

$$=\sum_{n=0}^{a-1} z^{-n}\left(x[n-c]u[n-c]\right)+\sum_{n=c}^{\infty} z^{-n}\left(x[n-c]u[n-c]\right)$$

From (2.8) $u[n-c]$=0 for $n < c$, which yields

$$Z\,[x[n-c]u[n-c]] = \sum_{n=c}^{\infty} x[n-c]z^{-n} \tag{2.10}$$

By taking $n-c=p$, (2.10)

$$Z\,[x[n-c]u[n-c]]=\sum_{p=0}^{\infty} z^{-(p+c)}x[p]=z^{-c}\sum_{p=0}^{\infty} x[p]z^{-p}$$

$$Z\,[x[n-c]u[n-c]]=Z\left[x[n]\right] \tag{2.11}$$

This shows the negative increment of 'c' units at power of variable z.
Similarly, for a function defined as $x[n-c]u[n]$, also delayed by 'c' units,
Now applying Z-transform on $x[n-c]u[n]$

$$Z[x[n\text{-}c]\;u[n]]=\sum_{n=0}^{\infty} z^{-n}\left(x[n-c]u[n]\right)=\sum_{n=0}^{c-1} z^{-n}\left(x[n-c]u[n]\right)+\sum_{n=c}^{\infty} z^{-n}\left(x[n-c]u[n]\right)$$

By (2.11) we have $u[n]$=0 for $n < c$, so we get

$$Z\,[x[n-c]u[n]]=\sum_{n=0}^{\infty} z^{-n}\left(x[n-c]u[n]\right)$$

Let $n-c=p$

$$Z\,[x[n-c]u[n]]= \sum_{p=-c}^{\infty} z^{-(p+c)}x[p] = z^{-c}\sum_{p=-c}^{\infty} z^{-p}x[p]$$

Now to start the summation at 0, we subtract the first terms then add these terms again

$$Z\,[x[n-c]u[n]] = z^{-c}\sum_{p=-c}^{\infty} z^{-p}x[p] - z^{-c}\sum_{p=-c}^{-1} z^{-p}x[p] + z^{c}\sum_{p=-c}^{-1} z^{-p}x[p]$$

$$= z^{-c}\sum_{p=1}^{c} z^{p}x[-p] + z^{-c}\sum_{p=0}^{\infty} z^{-p}x[p] = z^{-c}\sum_{p=1}^{c} z^{p}x[-p] + z^{-c}\,X(z)$$

Thus,

$$Z\,[x[n-c]u[n]]= z^{-c}\,X(z)+ z^{-c}\sum_{p=1}^{c} z^{p}x[-p]$$

2.9.4 Time Advance

Consider $x[n+c]u[n]$,i.e., advanced 'c' units, c>0. Also, truncate sequence at n=0.

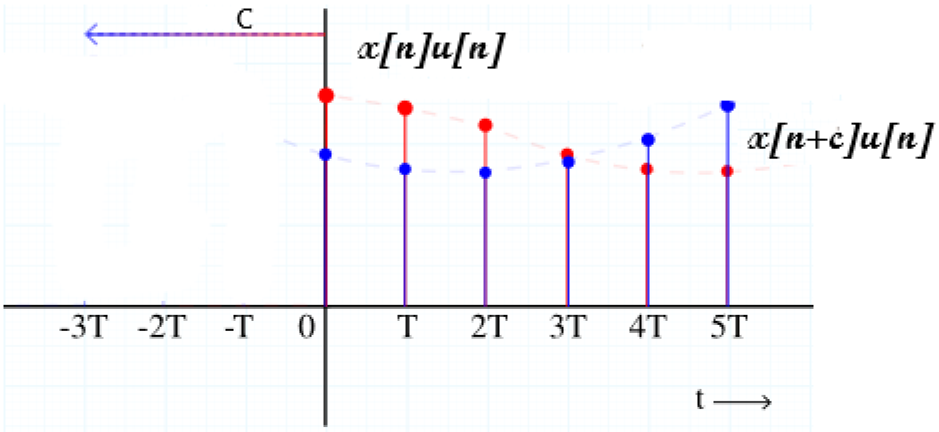

Fig 2.8: time advanced by 'c' units

Here $u[n]$ is unit-step function, given by;

$$u[n]=\begin{cases}0, & n<0\\ 1, & n>0\end{cases} \tag{2.12}$$

Applying transformation

$$Z\,[x[n+c]u[n]]=\sum_{n=0}^{c-1} z^{-n}\left(x[n+c]u[n]\right)+\sum_{n=c}^{\infty} z^{-n}\left(x[n+c]u[n]\right) \tag{2.13}$$

In first term of (2.13) replace n by $n+c$

$$Z\left[x[n+c]u[n]\right]=\sum_{n=c}^{\infty} z^{-n}\left(x[n+c]u[n]\right)+\sum_{n=-c}^{c-1} z^{-n-c}\left(x[n+c]u[n]\right)$$

From (2.8)

$$Z\,[x[n+a]u[n]] = \sum_{n=c}^{\infty} z^{-n}x[n+c] = z^{c}\sum_{p=0}^{c-1} z^{-p}x[p],$$

where $p=n+c$

$$Z\,[x[n+c]u[n]]=- z^{c}\sum_{p=0}^{c-1} z^{-p}x[p] + z^{c}\sum_{p=0}^{\infty} z^{-p}x[p]$$

This implies

$$Z\,[x[n+c]u[n]] = -z^{c}\sum_{p=0}^{c-1} z^{-p}x[p] + z^{c}X(z) \tag{2.14}$$

2.9.5 Modulation

We consider a signal of form $a^{n}y[n]u[n]$, multiplied by a scalar of form a^{n}, a>0 and truncate the sequence at. Now applying Z-transform

$$Z\,[a^{n}\;y[n]u[n]] = \sum_{n=0}^{\infty} z^{-n}\left(a^{n}y[n]u[n]\right)$$

$$Z\,[a^{n}\;y[n]u[n]] = \sum_{n=0}^{\infty}(a^{-1}z)^{-n}\,y[n] \tag{2.15}$$

Let $r = a^{-1}z$, now from the power series as,

$$\sum_{n=0}^{\infty} r^{n} = \frac{1}{1-r},\; |r|<1 \tag{2.16}$$

By using (2.16) in (2.15), we get

$$Z\,[a^{n}\;y[n]u[n]] = \sum_{n=0}^{\infty}(a^{-1}z)^{-n}\,y[n],\;\; |a^{-1}z|<1$$

$$Z\,[a^{n}\;y[n]u[n]] = Y\left(\frac{z}{a}\right)$$

which shows that in domain of z, $Y\left(\frac{z}{a}\right)$ has zero for $z=0$ and pole for point $z=a$.

2.9.6 Convolution

Convolution defines the multiplication of any two sequences in a complex domain. Advantage of Convolution is that it figures out an output sequence by the help of input signal and a transfer function, both of which are given in the time domain.

Considering convolution between two functions that are *s[p]* and *t[p]*, both are truncated at $p=0$. Then their convolution is given as

$$(s.t)[p]u[p] = \sum_{q=-\infty}^{\infty} s[q]t[p-q]u[p] \tag{2.17}$$

By the help of (2.12) we write (2.17) as,

$$(s.t)[p]u[p] = \sum_{q=-\infty}^{\infty} s[q]t[p-q]$$

Now apply Z-transform as

$$Z\left[(s.t)[p]u[p]\right] = \sum_{p=0}^{\infty}\left(z^{-p}\sum_{q=0}^{\infty}s[q]t[p-q]\right) = \sum_{q=0}^{\infty}\left(z^{-p}\sum_{p=0}^{\infty}s[q]t[p-q]\right)$$

As $s[q]$ is independent of p so we can take it out of summation as

$$Z\left[(x.y)[n]u[n]\right]=\sum_{m=0}^{\infty}\left(x[m]\sum_{n=0}^{\infty}z^{-n}y[n-m]\right)$$

From (2.12),

$$Z\left[(s.t)[p]u[p]\right]=\sum_{q=0}^{\infty}\left(s[q]z^{-q}\,T(z)\right)=T(z)\sum_{q=0}^{\infty}\left(s[q]z^{-q}\right)$$

$$Z[(s.t)[p]u[p]=S(z)T(z)$$

2.9.7 Conjugation

Consider a sequence with conjugation and is truncated at n=0 as s*[n]. Applying Z-transform

$$Z\left[s^*[n]\right]=\sum_{n=0}^{\infty}z^{-n}\,s^*[n]=\sum_{n=0}^{\infty}\left(s[n]\left(\frac{1}{(z^*)^n}\right)\right)^*=\left(\sum_{n=0}^{\infty}s[n]\left(\frac{1}{(z^*)^n}\right)\right)^*$$

This implies

$$Z\left[s^*[n]\right]=S^*(z^*).$$

2.9.8 First Difference

Taking difference in a discrete time signal is likewise taking differential in a continues signal. Suppose we have difference $y[p]-y[p-1]$ of two sequences $y[p]$ and $y[p-1]$. Apply Z-transform,

$$Z\left[y[p]-y[p-1]\right]=\sum_{p=0}^{\infty}z^{-p}y[p]-\sum_{p=0}^{\infty}z^{-p}y[p-1]$$

From (2.12) we get,

$$Z\left[y[p]-y[p-1]\right]=Y(z)-\left(\frac{1}{z}\right)Y(z)$$

$$Z\left[y[p]-y[p-1]\right]=Y(z)\left(1-\frac{1}{z}\right)$$

So, it changes the power of *z* variable respectively and subtracts it from units' difference of both functions.

2.9.9 Impulse

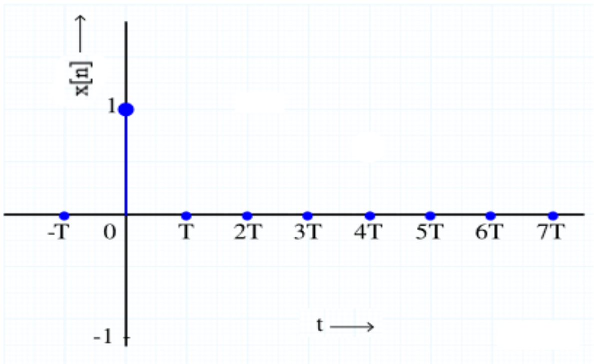

Fig 2.9: Discrete Impulse Signal

To study the pattern or behaviour of any discrete-time signal we analyse it through the impulse but in sense of discrete signals. Such an impulse is given by,

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$$x[n]=\begin{cases}1, n=0\\0, n\neq 0\end{cases}$$

Apply Z-transform on x[n]

$$X(z)=\sum_{n=1}^{\infty} z^{-n}x[n]+1$$

As the impulse is only generated at $n=0$ and it is 0 everywhere else, so
$X(z)=1$,
which is a similar form of Laplace transform of any continues impulse function.

2.9.10 n-Scaled
Consider a discrete-time function as previous one

$$x[n]=\begin{cases}0, n<0\\na^n, n\geq 0\end{cases}$$

Apply Z-transform and use unit step function given in (2.9)

$$Z[na^nu[n]]=\sum_{n=0}^{\infty} na^n z^{-n}$$

$$X(z)=az^{-1}+2a^2z^{-2}+3a^3z^{-3}+\ldots \tag{2.18}$$

By multiply both sides of (2.18) by az^{-1}, we get

$$az^{-1}X(z)=a^2z^{-2}+2a^3z^{-3}+3a^4z^{-4} \tag{2.19}$$

Now we subtract (2.19) from (2.18)

$$X(z)-az^{-1}X(z)=\left(az^{-1}+2a^2z^{-2}+3a^3z^{-3}+\ldots\right)-\left(a^2z^{-2}+2a^3z^{-3}+3a^4z^{-4}\ldots\right)$$

$$=a\left(\frac{1}{z}\right)+a^2\left(\frac{1}{z^2}\right)+a^3\left(\frac{1}{z^3}\right)+a^4\left(\frac{1}{z^4}\right)+\ldots$$

$$X(z)\left(1-\frac{a}{z}\right)=\frac{a}{z}\left(1+a\left(\frac{1}{z}\right)+a^2\left(\frac{1}{z^2}\right)+\ldots\right)=\left(\frac{a}{z}\right)\left(\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n\right)$$

$$X(z)=\frac{\left(\frac{a}{z}\right)}{\left(1-\left(\frac{a}{z}\right)\right)^2}$$

It follows that

$$Z[n\,a^nu[n]]=\frac{az}{(z-a)^2} \quad \text{, where } |z|>a \tag{2.20}$$

2.9.11 Exponential

$$f[n] = \begin{cases} 0, & n < 0 \\ e^{-anT}, & n \geq 0 \end{cases}$$

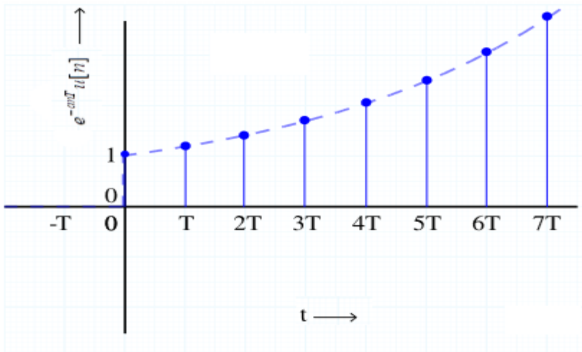

Fig 2.10: Discrete exponential function

Consider an exponential as well as discrete-time function $e^{-anT}u[n]$ and start the variation from $n = 0$. Apply Z-transform along with unit step function

$$X(z) = Z\left[e^{-anT}u[n]\right] = \sum_{n=0}^{\infty} e^{-anT} z^{-n} = \sum_{n=0}^{\infty}\left(e^{-aT}z^{-1}\right)^n$$

$$X(z) = \frac{z}{z - e^{-aT}}, \ |z| > e^{-aT}$$

2.9.12 Cosine

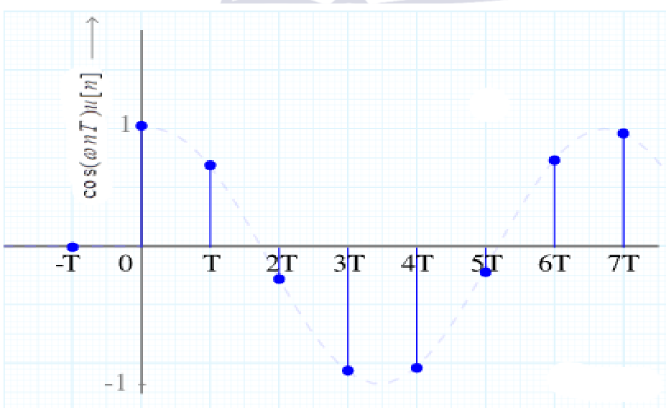

Fig 2.11: Discrete Cosine function

Consider the Cosine function $\cos(\omega nT)u[n]$ along with the unite step function and start the variation from $n = 0$ .Where, ω is the frequency and ωT is the initial frequency.

Now apply the Z-transform along with the definition of unit step function

$$X(z) = Z\left[\cos(\omega nT)u[n]\right]$$

$$= \sum_{n=0}^{\infty} \cos(\omega nT).z^{-n}$$

Use here the exponent form of cosine function as

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

This implies

$$X(z) = \sum_{n=0}^{\infty} z^{-n}\left(\frac{e^{i\omega nT} + e^{-i\omega nT}}{2}\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} z^{-n}e^{i\omega nT} + \sum_{n=0}^{\infty} z^{-n}e^{-i\omega nT}\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty}\left(z^{-1}e^{i\omega T}\right)^{n} + \sum_{n=0}^{\infty}\left(z^{-1}e^{-i\omega T}\right)^{n}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1-z^{-1}e^{i\omega T}} + \frac{1}{1-z^{-1}e^{-i\omega T}}\right), |z| > \left|e^{i\omega T}\right| = 1$$

$$= \frac{1}{2}\left(\frac{z}{z-e^{i\omega T}} + \frac{z}{z-e^{-i\omega T}}\right), |z| > 1$$

$$X(z) = \frac{z^2 - z\cos(\omega T)}{z^2 - 2i\sin(\omega T) + 1}, \; |z| > 1$$

2.9.13 Sine

The Z-transform of Sine function $\sin(\omega nT)u[n]$ is similar that of Cosine function.

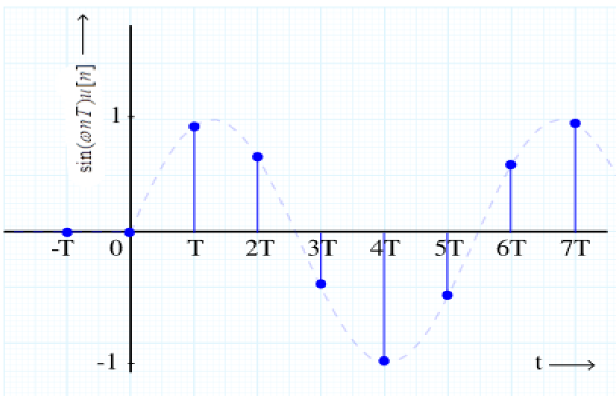

Fig 2.12: Discrete-time Sine function

We use the exponential form of sine function here as $\sin\theta = \dfrac{e^{i\omega nT} - e^{-i\omega nT}}{2i}$

Then,

$$X(z) = \sum_{n=0}^{\infty}\left(\frac{e^{i\omega nT} - e^{-i\omega nT}}{2i}\right)\left(\frac{1}{z^n}\right)$$

$$=\frac{1}{2i}\left(\sum_{n=0}^{\infty}z^{-n}e^{i\omega nT}-\sum_{n=0}^{\infty}z^{-n}e^{-i\omega nT}\right)$$

$$=\frac{1}{2i}\left(\sum_{n=0}^{\infty}\left(z^{-1}e^{i\omega T}\right)^{n}-\sum_{n=0}^{\infty}\left(z^{-1}e^{-i\omega T}\right)^{n}\right)$$

$$=\frac{1}{2i}\left(\frac{1}{1-z^{-1}e^{i\omega T}}-\frac{1}{1-z^{-1}e^{-i\omega T}}\right),\ |z|>\left|e^{i\omega T}\right|=1$$

$$=\frac{1}{2i}\left(\frac{z}{z-e^{i\omega T}}-\frac{z}{z-e^{-i\omega T}}\right),|z|>1$$

$$X(z)=\frac{Z\sin(\omega T)}{z^{2}-2zi\sin(\omega T)+1}\ ,\ |z|>1$$

2.9.14 Initial Value Theorem

If T(z) represents Z-transformation of t[n]. Then, $t[0]=\lim_{z\to\infty}T(z)\ \ and\ \ T(z)=\lim_{z\to 0}\left(\frac{1}{z}\right)$

Proof

Consider a discrete time function as $t[n]$ where $n<0$.

$$T(z)=\sum_{n=-\infty}^{\infty}t[n]\left(\frac{1}{z^{n}}\right)$$

$$\lim_{z\to\infty}T(z)=\lim_{z\to\infty}\sum_{n=-\infty}^{\infty}\frac{t[n]}{z^{n}}$$

$$=\sum_{n=-\infty}^{\infty}t[n]\lim_{z\to\infty}\left(\frac{1}{z^{n}}\right)$$

$$=t[0]+\sum_{n=1}^{\infty}t[n]\lim_{z\to\infty}\left(\frac{1}{z^{n}}\right)$$

This implies

$$t[0]=\lim_{z\to\infty}T(z)\,.$$

2.9.16 Final Value theorem

If *T(z)* shows Z-transform of *t[n]*. Then, $t[\infty]=\lim_{z\to 1}(z-1)T(z)$. Consider difference of a discrete time function with its own time advance function and take the delay of 1 step only

$$t[n+1]-t[n] \tag{2.21}$$

Transformation gives

$$Z\left[t[n+1]-t[n]\right]=\sum_{n=0}^{\infty}\left(\frac{t[n+1]-t[n]}{z^{n}}\right)$$

Applying limit for $z\to 1$

$$\lim_{z\to 1}\left(Z\left[t[n-1]-t[n]\right]\right)=\lim_{z\to 1}\left(\sum_{n=0}^{\infty}\left(\frac{t[n+1]-t[n]}{z^n}\right)\right)=\sum_{n=0}^{\infty}\left(t[n+1]-t[n]\right)\lim_{z\to 1}\left(\frac{1}{z^n}\right)$$

$$\lim_{z\to 1}\left(Z\left[t[n-1]-t[n]\right]\right)=\lim_{n\to\infty}\left(t[1]+t[2]+\ldots+t[n-1]+t[n]+t[n+1]-t[0]-t[1]-t[2]-\ldots-t[n-1]-t[n]\right)$$

$$=\lim_{n\to\infty}t[n]-t[0] \qquad (2.22)$$

Apply the Z-transform on (2.21) but this time utilize the time delay property. Also applying limit for $z\to 1$

$$\lim_{z\to 1}Z\left[t[n+1]-t[n]\right]=\lim_{z\to 1}\left(z\left(T(z)-t[0]\right)-T(z)\right)$$

$$=\lim_{z\to 1}\left(zT(z)-zt[0]-T(z)\right)$$

$$=\lim_{z\to 1}\left(T(z)(z-1)\right)-\lim_{z\to 1}zt[0]$$

$$\lim_{z\to 1}Z\left[t[n+1]-t[n]\right]=\lim_{z\to 1}\left(T(z)(z-1)\right)-t[0] \qquad (2.23)$$

Now equate equation (2.22) and (2.23)

$$\lim_{z\to 1}\left(T(z)(z-1)\right)-t[0]=\lim_{n\to\infty}t[n]-t[0]$$

$$\lim_{n\to\infty}t[n]=\lim_{z\to 1}\left((z-1)T(z)\right)$$

which shows that for the final value we will require the transformed function of given discrete signal.

Chapter-3

3.1 Z-transform and Difference Equations

3.1.1 Definition

An equation that is expressed in the terms of a function of one or more than one independent variable along with the finite differences of the function, is called a difference equation.

A difference equation depicts parallel discrete form of differential equation, as it contains finite or discrete differences instead of derivatives.

3.2 Solution of famous difference equations

3.2.1 Fibonacci Recursion Expression

$$F_n=F_{n-1}+F_{n-2},\ F_0=0,\ F_1=1.$$

If F_n is represented by $y(n)$, then $y(1)=1\ and\ y(0)=0$,that gives the following

$$y(n)=y(-1+n)+y(-2+n) \qquad (3.1)$$

Applying Z-transform on (3.1), we obtain

$$Z[y(n)]=Z[y(-1+n)]+Z[y(-2+n)]$$

$$Y(z)=y(-2)+y(-1)+\frac{Y(z)}{z^2}+\frac{Y(z)+y(-1)}{z} \qquad (3.2)$$

By using the initial conditions in (3.1), we get

$$y(-2)+y(-1)=0 \qquad (3.3)$$

$$y(-1)=1 \qquad (3.4)$$

Using (3.3) and (3.4), we get

$$y(-1)=1\ and\ y(-2)=-1 \qquad (3.5)$$

using (3.5), (3.2), gives

$$Y(z)=\frac{Y(z)}{z^2}+\frac{1}{z}\left(Y(z)+1\right)$$

$$Y(z)=Y(z)\left(\frac{1}{z}+\frac{1}{z^2}\right)+\frac{1}{z}$$

$$Y(z)\left(1-\frac{1}{z}-\frac{1}{z^2}\right)=\frac{1}{z}$$

$$Y(z)=\frac{\left(\frac{1}{z}\right)}{(z^2-z-1)\left(\frac{1}{z^2}\right)}$$

$$Y(z)=\frac{z}{-1-z+z^2}$$

From denominator we have $z=1+\frac{\sqrt{5}}{2}, z=1-\frac{\sqrt{5}}{2}$

Let a=$1+\frac{\sqrt{5}}{2}$ and b=$1-\frac{\sqrt{5}}{2}$ then,

$$Y(z)=\frac{z}{(z-a)(z-b)}$$

By partial fraction we have,

$$Y(z)=\left(a-b\right)^{-1}\left(-\frac{b}{z-b}+\frac{a}{z-a}\right)$$

Here the ROC is determined to be $|z|>a$,where $z=a$ and $z=b$ are the simple poles. Now apply the Z-inverse transform as,

$$Z^{-1}\left[Y(z)\right]=Z^{-1}\left[\left(a-b\right)^{-1}\left(-\frac{b}{z-b}+\frac{a}{z-a}\right)\right]$$

$$y(n)=\left(a\lim_{z\to a}(z-a)\frac{z^{n-1}}{(z-a)}-b\lim_{z\to a}(z-b)\frac{z^{n-1}}{(z-b)}\right)\frac{1}{a-b}$$

$$y(n)=\left(a.a^{n-1}-b.b^{n-1}\right)\frac{1}{a-b}=\frac{a^n-b^n}{a-b}$$

$$F_n=\frac{a^n-b^n}{\sqrt{5}},\ n\geq 0,$$

where $a=1+\frac{\sqrt{5}}{2}$ and b=$1-\frac{\sqrt{5}}{2}$.

above is the compact form of Fibonacci Recursion relation.

3.2.2 Chebyshev's Polynomials

Consider the recursion relation of Chebyshev's polynomials as

$$P_n(x) = 2xP_{n-1}(x) - P_{n-2}(x) \tag{3.6}$$

$$P_0(x) = 1,\ P_1(x) = x. \tag{3.7}$$

Let us replace $T_n(x)$ by $y(n)$. Then (3.6) and (3.7) become

$$y(n) = -y(-2+n) + 2x\,y(-1+n), \tag{3.8}$$

$$y(0) = 1,\ y(1) = x. \tag{3.9}$$

Applying Z-transform, we have

$$Y(z) = 2x\left(z^{-1}Y(z) + y(-1)\right) - \left(z^{-2}Y(z) + y(-2) + z^{-1}y(-1)\right)$$

By using (3.9) in (3.8),

$$y(-1) = x,$$

$$y(-2) = 2x^2 - 1.$$

This implies

$$Y(z) = 2x\left(x + \frac{Y(z)}{z}\right) - \left(2x^2 - 1 + \frac{Y(z)}{z^2} + \frac{x}{z}\right)$$

$$Y(z) = \frac{(z-x)z}{1 - 2xz + z^2}$$

$$Y(z) = \frac{z(z-x)}{(z-\alpha)(z-\beta)},\quad \alpha = x + \sqrt{x^2-1} \text{ and } \beta = x - \sqrt{x^2-1}.$$

Applying inverse Z-transform as

$$Z^{-1}\left[Y(z)\right] = Z^{-1}\left[\frac{z(z-x)}{(z-\alpha)(z-\beta)}\right]$$

Here α and β are the simple poles so,

$$y(n) = Residue(Y(z)z^{n-1}, \alpha) + Residue(Y(z)z^{n-1}, \beta)$$

$$y(n) = \lim_{z\to\beta}\frac{z^n(-x+z)}{(-\alpha+z)} + \lim_{z\to\alpha}\frac{z^n(-x+z)}{(-\beta+z)}$$

$$y(n) = \frac{(\alpha-x)\alpha^n - (\beta-x)\beta^n}{(\alpha-\beta)}$$

As we know that $x = \cos\theta$, implies that $\alpha = e^{i\theta}$ *and* $\beta = e^{-i\theta}$, which gives,

$$y(n) = \frac{1}{2}\left(e^{in\theta} + e^{-in\theta}\right)$$

$$y(n) = \cos\left(n\theta\right)$$

$$P_n(\cos\theta) = \cos\left(n\theta\right),\ n \ge 0.$$

Above is the compact form of Chebyshev's recursion relation.

3.2.3 Legendre Difference Equation

The Legendre Polynomial $p_n(x)$ gives difference equation as

$$(1+n)p_{n+1}(x)-(2n+1)xp_n(x)+(n-1)p_{n-1}(x)=0\,, \tag{3.10}$$

with $p_0(x)=1 \; and \; p_1(x)=x$.

Let $y(n)=p_n(x)$ express the nth term in (3.10) as

$$n\,y(n)=(2n-1)\,x\,y(n-1)-(n-1)\,y(n-2),$$

By using $y(0)=p_0(x)=1 \, and \; y(1)=p_1(x)=x$, we obtain

$$y(-2)=x \, and \; y(-1)=1 \tag{3.11}$$

Apply Z-transform on (3.10)

$$\sum_{n=0}^{\infty}\frac{n\,y(n)}{z^n}=\sum_{n=0}^{\infty}\frac{x(-1+2n)\;y(-1+n)}{z^n}-\sum_{n=0}^{\infty}\frac{(-1+n)\,y(-2+n)}{z^n}$$

$$-zY(z)=-x\left[-y(-1)+2Y'(z)-Y(z)z^{-1}+y(-1)z^{-2}\right]+y(-2)+Y'(z)z^{-1}-Y(z)z^{-2}$$

By using (3.11), we have following simplified form

$$\left(-z+2z-\frac{1}{z}\right)Y'(z)+\left(\frac{1}{z^2}-\frac{x}{z}\right)Y(z)=0 \tag{3.12}$$

After simplification, equation (3.12) gives

$$Y(z)=\frac{az}{\sqrt{z^2-2xz+1}} \tag{3.13}$$

To find the value of a, substitute $y(0)=1$ in equation (3.13),

$$\sum_{n=0}^{\infty}z^{-2n-1}z^{n}=\sum_{n=0}^{\infty}az^{n}$$

Comparing coefficient of z^n , we obtain

$$a=z^{-2n-1} \tag{3.14}$$

Substitute (3.14) in (3.13) and simplify,

$$Y(z)=\frac{z^{-2n}}{\sqrt{z^2-2xz+1}}$$

Applying inverse *z*-transform and theorem 2.5, we obtain

$$y(n)=\frac{1}{2\pi i}\oint_c\frac{\left(\frac{1}{z^{n+1}}\right)}{\sqrt{1-2xz+z^2}}dz\,,$$

where c is any positive oriented closed contour surrounding at the point *z*=0 and lying inside the disk $|z|<r$ .Ultimately, we can write Legendre polynomial $p_n(x)=y[n]$ in the form given below

$$p_n(x)=\frac{1}{2\pi i}\oint_c\frac{z^{-n-1}}{\sqrt{z^2-2xz+1}}dz,$$

Preceding equation is the solution for (3.10).

3.2.4 Second Order Deterministic Difference Equation

Consider $t(n) = 3 + 0.9t(-1+n) - 0.2t(-2+n)$ (3.14)

$$t(0) = 0, t(1) = 2. \tag{3.15}$$

By using (3.15) in (3.14), we have

$$3 + 0.9t(-1) - 0.2t(-2) = 0 \tag{3.16}$$

$$3 + 0.9t(0) - 0.2t(-1) = 2 \tag{3.17}$$

by solving (3.16) and (3.17) simultaneously, we have

$$t(-1) = 5 \tag{3.18}$$

$$t(-2) = 37.5 \tag{3.19}$$

Applying Z-transform on (3.14), we get

$$T(z) = \sum_{n=0}^{\infty} \frac{3}{z^n} + 0.9\left(t(-1) + \frac{T(z)}{z}\right) - 0.2\left(t(-2) + \frac{t(-1)}{z} + \frac{T(z)}{z^2}\right)$$

After simplification, we obtain

$$T(z) = \frac{2z\left(z + \frac{1}{2}\right)}{\left(0.2 - 0.9z + z^2\right)(z-1)}$$

Apply inverse transformation, we have

$$t(n) = Z^{-1}\left(\frac{2z\left(z+\frac{1}{2}\right)}{(-1+z)(-0.4+z)(-0.5+z)}\right)$$

$$t(n) = Residue(T(z)z^{n-1}, 1) + Residue(T(z)z^{n-1}, 0.4) + Residue(T(z)z^{n-1}, 0.5)$$

That implies

$$t(n) = 10 - 40(0.5)^n - 30(0.4)^n.$$

Conclusion

This paper presented a detailed study of the Z-transform as an effective analytical framework for discrete-time signals and systems. The fundamental concepts of the Z-transform, including its definition, Region of Convergence (ROC), and essential properties, were thoroughly investigated. The ROC was shown to play a central role in determining the existence, stability, and causality of discrete-time systems, thereby providing valuable insight into system behavior in the complex domain.

Particular emphasis was placed on the inverse Z-transform and its computation through the Cauchy Residue Theorem. By employing complex variable techniques, inverse transformations were obtained in a systematic and mathematically rigorous manner. The residue approach proved to be a powerful tool for recovering discrete-time sequences from their transformed representations and for simplifying the analysis of systems described by difference equations.

The applicability of the developed theory was demonstrated through the solution of several important recursive relations and difference equations. Compact closed-form expressions were obtained for the Fibonacci recursion, Chebyshev polynomials, Legendre difference equations, and second-order deterministic difference equations. These examples illustrate the effectiveness of the Z-transform in converting complex recurrence relations into algebraic expressions that can be solved efficiently in the transformed domain.

The results of this study confirm that the Z-transform is one of the most versatile tools for the analysis of discrete-time systems. Its ability to characterize system properties, simplify computations, and generate compact solutions to recursive relations makes it indispensable in mathematics, signal processing, communication systems, and control theory. Future work may focus on extending these techniques to nonlinear difference equations, multidimensional discrete systems, and modern applications in digital signal processing and computational mathematics.

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