

Optimization Strategies for Image Registration using Rigid and Möbius Groups

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Abstract

Image registration is an important technique used to align two or more images by minimizing the mismatch between their visual appearances. It has wide applications in areas such as medical imaging, computer vision, remote sensing, and biological shape analysis. In this study, optimization strategies for two-dimensional image registration are investigated using rigid and Möbius transformations. The rigid transformation model consists of scaling, rotation, and translation parameters, while the Möbius transformation provides a more flexible framework for handling complex geometric deformations.

The registration process is formulated through a discretized least-squares objective function based on the sum of squared differences between pixel intensities of the source and target images. To obtain accurate alignment, different optimization strategies are explored, including coarse search, gradient descent, and landmark-based initialization. Coarse search is used to explore the transformation space and provide an approximate solution, whereas gradient descent refines the transformation parameters to achieve improved registration accuracy. In addition, landmark initialization is incorporated to generate reliable initial estimates for difficult registration cases and to reduce the possibility of convergence toward local minima.

The proposed framework is theoretically analyzed to evaluate the effectiveness of the optimization strategies. The analysis demonstrates that the combined use of coarse search, landmark initialization, and gradient-based refinement improves image alignment and optimization performance. This study highlights the effectiveness of combining rigid and Möbius transformation frameworks within a unified optimization-based image registration approach.

1. INTRODUCTION

Image registration is the process of aligning two or more images so that their visual

structures and geometric features correspond within a common coordinate system. It is an important technique in computer vision,

medical imaging, remote sensing, biological analysis, and pattern recognition because it allows meaningful comparison between multiple datasets [1-4]. In two-dimensional image registration, the primary objective is to determine an optimal transformation that minimizes the mismatch between a source image and a target image. The quality of registration depends on both the selected transformation model and the optimization strategy used during alignment [5].

Different transformation models have been proposed for image registration depending on the complexity of deformation between images. Among these models, rigid transformations are widely used because they preserve the structural properties of objects through scaling, rotation, and translation operations [5,6]. Rigid registration is computationally efficient and performs effectively when the geometric deformation between images is relatively small. However, in many practical applications, rigid transformations alone may not be sufficient to model complex nonlinear deformations or shape variations [2].

To overcome these limitations, more flexible transformation frameworks have been investigated. Möbius transformations belong to the family of conformal mappings and provide a generalized approach for geometric image deformation [2, 7]. These transformations combine scaling, rotation, translation, and inversion within a unified mathematical framework while preserving local angular structures. Due to these properties, Möbius transformations have gained attention in image analysis and shape mapping problems where smooth deformations are required [2].

Optimization plays a central role in image registration because the transformation parameters are obtained by minimizing an objective function that measures the discrepancy between images [5, 8]. In many registration problems, the optimization landscape contains several local minima, making the estimation process difficult.

Therefore, robust optimization techniques are required to improve convergence and alignment accuracy. In this study, coarse search, gradient descent, and landmark-based initialization methods are investigated for rigid and Möbius image registration. The coarse search method explores the transformation space to provide approximate parameter estimates, while gradient descent refines these estimates to achieve accurate alignment. In addition, landmark initialization is incorporated to generate reliable starting points for challenging registration cases [2, 5].

The main objective of this paper is to investigate optimization strategies for image registration using rigid and Möbius transformations within a unified framework. Numerical experiments are performed on both synthetic and real image datasets to evaluate the effectiveness of the proposed methods. The obtained results demonstrate that combining coarse search, landmark initialization, and gradient-based refinement improves registration accuracy and reduces alignment errors in difficult optimization scenarios.

2. IMAGE REGISTRATION FRAMEWORK

Image registration is a process in which one image is transformed in such a way that it aligns with another image under a common coordinate system. In most registration problems, the image to be transformed is called the source image, while the reference image is called the target image. The objective of registration is to determine a transformation that minimizes the difference between these two images [1, 2]. Let that I_1 and I_2 denote the source and target grayscale images defined on a domain $\Omega \subset \mathbb{R}^2$. The registration problem can be formulated as an optimization problem in which a transformation ϕ is determined by minimizing an objective function based on image intensity differences. A commonly used similarity measure is the Sum of Squared Differences (SSD), defined as follows [2, 3]:

$$E_{I_1, I_2}(\phi) = \int_{\Omega} \|I_1 \circ \phi^{-1}(x_1, x_2) - I_2(x_1, x_2)\|^2 dx_1 dx_2,$$

Where ϕ^{-1} represents the inverse transformation applied to the source image. The objective function measures the discrepancy between the transformed source image and the target image. The optimal registration is obtained when the value of the objective function is minimized [2].

For numerical implementation, the continuous formulation of the objective function is discretized over a finite image grid. Let K denote the discrete computational domain containing $L \times L$ grid points see in Figure 2.1. The discrete form of the SSD objective function is expressed as follows [5]:

$$E(\phi) = \sum_{i=1}^L \sum_{j=1}^L \{(I_1 \circ \phi^{-1})(x_{ij}) - I_2\}^2$$

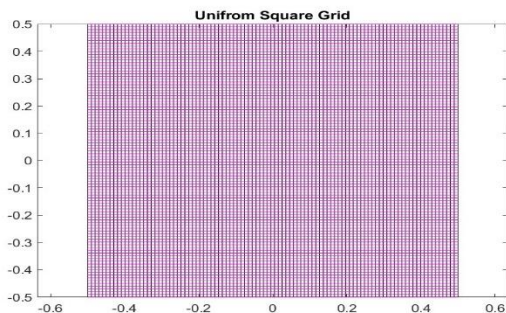


Figure 2.1: The grid domain has 10,000 node points in $[-0.5, 0.5] \times [-0.5, 0.5]$ range along with $L = 100$ (there are 100 node points on each side).

In practical image registration problems, interpolation is required because the transformed coordinates generally do not coincide exactly with grid points. Bilinear interpolation is commonly used to estimate the intensity values at transformed locations. If transformed points fall outside the computational domain, a constant background intensity is assigned during the

optimization process [5]. The effectiveness of image registration strongly depends on the selected transformation model and optimization method. In this study, rigid and Möbius transformations are investigated together with different optimization strategies in order to improve registration accuracy and convergence performance.

3. RIGID TRANSFORMATION FOR IMAGE REGISTRATION

Rotation

Suppose an arbitrary $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. Let $\theta \in [0, 2\pi)$ be the angle of rotation transformation ϕ_1 to generate the new point $\begin{pmatrix} x' \\ y' \end{pmatrix} \in \mathbb{R}^2$ as defined by:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \phi_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Scaling

Suppose an arbitrary $\lambda \in \mathbb{R}^2$ is a non-zero number so uniform scaling transformation ϕ_2 over $\begin{pmatrix} x \\ y \end{pmatrix}$ is defined below:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \phi_2 \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

Translation

Suppose $t_x, t_y \in \mathbb{R}$ are the translations in the horizontal and vertical axis respectively. The translation transformation ϕ_3 over $\begin{pmatrix} x \\ y \end{pmatrix}$ is defined below:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \phi_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \end{pmatrix}$$

The rigid transformation is the combination of the all above mentioned transformations. The compact form of the rigid transformation ϕ is defined below:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \phi \begin{pmatrix} x \\ y \end{pmatrix} = (\phi_3 \circ \phi_2 \circ \phi_1) \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \lambda \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix},$$

$$\theta \in [0, 2\pi), 0 \neq \lambda \in \mathbb{R}$$

The collection of all such ϕ forms a group under composition, which is known as rigid

Group. (From Equation 2.1) The inverse of a rigid transformation is:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{\lambda} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} \left\{ \begin{pmatrix} x' \\ y' \end{pmatrix} - \begin{pmatrix} t_x \\ t_y \end{pmatrix} \right\} \\ &= \frac{1}{\lambda} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{-1} \left\{ \begin{pmatrix} x' \\ y' \end{pmatrix} - \begin{pmatrix} t_x \\ t_y \end{pmatrix} \right\} \end{aligned}$$

This transformation model forms a finite-dimensional group under composition and serves as the primary framework for rigid image registration. The parameters θ , λ , t_x , and t_y are determined by minimizing the L2-norm objective function discussed in the previous section [5].

The rigid registration framework is computationally efficient and suitable for many practical image alignment problems. However, in cases involving more complex geometric distortions, generalized transformations such as Möbius mappings may provide additional flexibility [2].

4. MÖBIUS TRANSFORMATION FOR IMAGE REGISTRATION

Although rigid transformations are effective for many image registration problems, they may become insufficient when the deformation between images is more complex. To address such cases, more flexible transformation models can be introduced. Among these models, Möbius transformations provide a generalized framework capable of representing scaling, rotation, translation, and inversion within a single mathematical formulation [2,7]. A Möbius transformation is defined on the complex plane and can be written as follows [2]:

$$\phi(z) = \frac{pz+q}{rz+s}$$

where $p, q, r, s \in \mathbb{C}$ and

$$ps - qr \neq 0$$

The transformation acts on a complex variable z and maps one region of the plane onto another.

Möbius transformations belong to the class of conformal mappings, meaning that they preserve local angular structures during deformation [2]. An important property of the Möbius framework is that rigid transformations can be considered a special case of Möbius transformations. When the parameter ($r = 0$), the transformation reduces to a combination of scaling, rotation, and translation operations [2]. Therefore, the rigid transformation model discussed previously is naturally contained within the Möbius transformation framework. For image registration, the transformation parameters are determined by minimizing the discrepancy between the transformed source image and the target image through the SSD objective function introduced earlier. Due to the increased flexibility of Möbius mappings, the optimization process may become more sensitive to the initial parameter estimates. Consequently, appropriate optimization strategies are required to improve convergence and avoid local minima [2,8]. In this study, Möbius transformations are incorporated as an extension of the rigid registration framework in order to investigate more flexible geometric alignment strategies. Rather than focusing on complex mathematical derivations, the emphasis is placed on optimization performance and practical image registration applications.

5. OPTIMIZATION TECHNIQUES

Optimization plays a fundamental role in image registration because the transformation parameters are obtained by minimizing the objective function that measures the discrepancy between the source and target images [4,8]. In practical registration problems, the optimization landscape may contain multiple local minima, making accurate alignment difficult. Therefore, efficient optimization strategies are required to improve convergence and registration accuracy [9]. In this study, three optimization strategies are investigated for rigid and Möbius image registration: coarse search, gradient descent, and landmark-based initialization.

5.1 Coarse Search Method

The coarse search method is a simple optimization approach in which the objective function is evaluated over a predefined set of transformation parameters [5]. Instead of relying on derivatives,

the method explores a large portion of the transformation domain and identifies approximate parameter values corresponding to low objective function values. For rigid registration, the search is performed over discrete values of rotation, scaling, and translation parameters. The objective function is evaluated repeatedly for different combinations of parameters, and the parameter set producing the minimum error is selected as an initial estimate [5]. One advantage of coarse search is that it provides a global overview of the optimization domain and reduces sensitivity to local minima. However, the computational cost increases significantly as the number of transformation parameters grows [5,8].

5.2 Gradient Descent Optimization

Gradient-based optimization methods are commonly used to refine the transformation parameters obtained from coarse search [4]. In this work, MATLAB's least-squares optimizer `lsqnonlin` is utilized to minimize the SSD objective function. The optimization process iteratively updates the transformation parameters in order to reduce the discrepancy between the transformed source image and the target image. The least-squares formulation can be expressed as follows [5]:

$$\min_x \|f(x)\|_2^2$$

where $f(x)$ represents the residual error between corresponding image intensities. Gradient descent methods are highly effective when a suitable initial guess is available. However, poor initialization may cause convergence toward local minima instead of the global optimum [4,8].

5.3 Landmark-Based Initialization

To improve optimization performance in difficult registration problems, landmark-based initialization is incorporated into the registration framework [2]. In this approach, corresponding feature points are selected manually or automatically from the source and target images. These landmarks provide an approximate

geometric relationship between the images and help generate reliable initial parameter estimates. For Möbius registration, the landmark approach is particularly useful because the transformation model contains additional flexibility compared to rigid transformations [2]. By using landmark correspondences before gradient refinement, the optimization process becomes more stable and less sensitive to initialization errors. The combination of coarse search, landmark initialization, and gradient-based refinement provides a robust optimization strategy for image registration. This combined framework improves convergence performance and enhances alignment accuracy for both rigid and Möbius transformation models.

6. COMBINED OPTIMIZATION STRATEGY

The accuracy of image registration strongly depends on the initialization and refinement stages of the optimization process. Individual optimization methods often perform well in specific situations, but they may fail when used independently in complex registration problems. For this reason, a combined optimization framework is proposed in this study by integrating coarse search, landmark initialization, and gradient-based refinement into a unified strategy. The first stage of the framework utilizes the coarse search method to explore the transformation domain and identify approximate parameter values [5]. Since coarse search evaluates the objective function over multiple parameter combinations, it provides a global overview of the optimization landscape and reduces the probability of selecting poor initial estimates. Although this method is computationally expensive, it is useful for obtaining a stable starting point for further optimization. In the second stage, landmark-based initialization is incorporated to improve the alignment between the source and target images [2]. Corresponding landmarks provide geometric guidance for the optimization process and help generate more accurate transformation estimates, particularly in difficult registration cases involving complex deformations. This step is especially important for Möbius registration

because the transformation model contains additional flexibility compared to rigid transformations. Finally, gradient descent optimization is applied to refine the transformation parameters obtained from the previous stages. The least-squares optimizer iteratively minimizes the SSD objective function and improves the alignment accuracy between the transformed source image and the target image [4,5]. Starting the optimization process from reliable initial estimates significantly improves convergence performance and reduces the risk of trapping in local minima. The proposed optimization pipeline can therefore be summarized as follows:

Coarse Search → Landmark Initialization → Gradient-Based Refinement

This combined strategy improves the robustness of the registration framework and provides more stable optimization performance for both rigid and Möbius transformation models.

7. OPTIMIZATION ANALYSIS and DISCUSSION

The optimization strategy plays an important role in image registration because the objective function may contain multiple local minima [8]. Rigid transformations provide a simple and computationally efficient framework for image alignment due to the limited number of transformation parameters [5]. However, rigid registration may not accurately model complex geometric deformations. Möbius transformations provide additional flexibility by incorporating scaling, rotation, translation, and inversion within a single framework [2]. This flexibility improves deformation handling but also increases optimization complexity. Therefore, reliable initialization becomes important for stable convergence. The combined use of coarse search, landmark initialization, and gradient refinement improves optimization performance. Coarse search provides approximate parameter estimates, landmark initialization improves stability, and gradient descent refines the final alignment [2,5]. The integration of these methods improves convergence behavior and reduces the possibility

of trapping in local minima.

8. ADVANTAGES AND LIMITAGES

The proposed framework combines rigid and Möbius transformations within a unified optimization-based registration model. The approach provides improved flexibility for handling both simple and complex image deformations. In addition, the use of combined optimization strategies improves convergence accuracy and optimization stability. However, Möbius transformations increase computational complexity compared to rigid registration. Similarly, coarse search methods may require large computational time when the transformation domain becomes large. The overall performance also depends on suitable initialization and parameter selection.

9. CONCLUSION

In this paper, optimization strategies for image registration using rigid and Möbius transformations were investigated. The registration process was formulated using an SSD-based objective function together with coarse search, gradient descent, and landmark-based initialization techniques. The study focused on developing a unified optimization framework capable of improving image alignment accuracy and optimization stability. Rigid transformations provided an efficient and computationally stable framework for handling simple image registration problems involving scaling, rotation, and translation. In contrast, Möbius transformations introduced additional flexibility for representing more complex geometric deformations within a generalized conformal mapping framework. The integration of these transformation models allowed the registration framework to address both simple and moderately complex alignment problems. Different optimization strategies were analyzed to improve convergence behavior during the registration process. Coarse search provided approximate parameter estimation, landmark-based initialization improved optimization stability, and gradient-based refinement enhanced alignment accuracy through iterative minimization of the SSD objective function. The combined optimization framework reduced sensitivity to local minima and improved the

overall robustness of the registration process. Overall, the study demonstrates that integrating multiple optimization strategies within a unified rigid and Möbius registration framework provides an effective approach for two-dimensional image registration problems. The proposed methodology offers a balance between computational efficiency, geometric flexibility, and optimization stability, making it suitable for various image registration applications.

REFERENCES

- [1] B. Zitová and J. Flusser, "Image registration methods: A survey," *Image and Vision Computing*, vol. 21, no. 11, pp. 977-1000, 2003.
- [2] M. Y. Tufail and S. Gul, "An Optimisation over the Möbius Group for an Optimal Solution in Image Registration," *Punjab University Journal of Mathematics*, vol. 57, no. 6, pp. 670-686, 2025.
- [3] L. G. Brown, "A Survey of Image Registration Techniques," *ACM Computing Surveys*, vol. 24, no. 4, pp. 325-376, 1992.
- [4] J. B. Antoine Maintz and M. A. Viergever, "A Survey of Medical Image Registration," *Medical Image Analysis*, vol. 2, no. 1, pp. 1-36, 1998.
- [5] M. Y. Tufail, "Image registration under conformal diffeomorphisms," Ph.D. dissertation, Massey University, Palmerston North, New Zealand, 2017. B. Fischer and J. Modersitzki, "Ill-Posed Medicine—An Introduction to Image Registration," *Inverse Problems*, vol. 24, no. 3, pp. 1-16, 2008.
- [6] T. Needham, *Visual Complex Analysis*. Oxford, U.K.: Oxford University Press, 1997.
- [7] J. Nocedal and S. Wright, *Numerical Optimization*. New York, NY, USA: Springer, 2006.
- [8] D. E. Rumelhart, G. E. Hinton, and R. J. Williams, "Learning Representations by Back-Propagating Errors," *Nature*, vol. 323, pp. 533-536, 1986.
- [9] K. V. Mardia and P. E. Jupp, *Directional Statistics*. Hoboken, NJ, USA: Wiley, 2000.
- [10] S. Baker and I. Matthews, "Lucas-Kanade 20 Years On: A Unifying Framework," *International Journal of Computer Vision*, vol. 56, no. 3, pp. 221-255, 2004.
- [11] J. Modersitzki, *Numerical Methods for Image Registration*. Oxford, U.K.: Oxford University Press, 2004.
- [12] R. Szeliski, *Computer Vision: Algorithms and Applications*. London, U.K.: Springer, 2010.
- [13] D. Forsyth and J. Ponce, *Computer Vision: A Modern Approach*. Upper Saddle River, NJ, USA: Pearson, 2011.

