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P3 CHROMATIC POLYNOMIALS OF DIFFERENT GRAPHS

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Abstract
This thesis explores the P_3-chromatic polynomial, a crucial exten-sion of the classical chromatic polynomial in graph theory, aimed at examining path-based coloring constraints. The focus of the research is on identifying the P_3-chromatic polynomials for a range of graph fam-ilies, including cycle graphs, path graphs, ladder graphs, star graphs, and wheel graphs. Utilizing core principles of vertex coloring, chromatic numbers, and P_3-chromatic numbers, this work systematically calcu-lates the P_3-chromatic polynomials for these graph categories. The results obtained shed light on the structural properties of graphs under P_3-coloring conditions, thereby enhancing the theoretical framework of graph theory.

1 Introduction

The geometric representation of objects and their complex interrelations is an es-sential language of mathematics, which can be described using graph theory. In this study, P_3- chromatic polynomials are discussed and their dimensions are ex-plored. In this regard, graph coloring is an area of deep interest in this field due to its extensive practice and its theoretical foundations [12]. The given study is based on the new extension of the classical vertex coloring, called P_3- coloring, and the study of the P_3-chromatic polynomial to analyze path-based constraints. To provide a solid background for the current study, one must first define the fundamental elements of algebraic graph theory, such as chromatic number and chromatic polynomial [28]. These measures enable us to determine the number of ways a graph can be colored without violating certain adjacent conditions. The given section presents these basic definitions and cyclic graph propositions, which form the foundational background for the proofs and discussions that follow. Ap-plying these classical ideas to the coloring of P 3 graphs, this work is expected to offer a more subtle insight into the structure of graphs: path, cycle, ladder, a star, and wheel graph.

Likewise, the discipline of graph theory started with Leonhard Euler [13] in the Eighteenth Century with Euler's paper when attempting to solve complicated geometric problem of Seven Bridges of Königsberg he formulated the notion of a graph to represent the network problems based on paths and joints [14]. During subsequent 200 years, the growth of graph theory was not very rapid, however, it was advanced by Augustin Cauchy [15], Gustav Kirchhoff [16], Arthur Cayley [17] etc. This work set the stage for graph theory to be recognized more formally, while other mathematicians also made early attempts at applying graph theory, most prominently in trees pertaining to chemistry with Arthur Cayley [18] or the first book on graph theory written by Dénes Kőnig [19]. These contributions proven graph theory as a tool for studying connections and relations and its uses extend to other disciplines including network science, biology, chemistry and computer engineering. First works on cycles and trees which were laid down by Otto Schreier in 1929 [20], on planar graphs by George Pólya in 1937 [21], and coloring problems by Philip Franklin in 1922 [22].

The significance of graph coloring can be traced back to 1852 with the four color problem posed by Francis Guthrie who stated that any map can be colored using no more than four colors ensuring that no two neighboring regions are colored identically [23]. Chromatic polynomial was defined by George Birkhoff in 1912 for counting the colorings on the graphs [24]. The chromatic polynomial $P(G, \lambda)$ enumerates the total number of valid λ-colorings for a graph G [25]. As mentioned above, computation of exact chromatic number or testing k-colorability is NP - complete but the chromatic polynomial contain the colorability information in a compressed form. The first method to evaluate $P(G, \lambda)$ which has been proposed was deletion and contraction. The deletion-contraction technique is more than just a technique: Whitney's innovation leads to this development [26]. This recursive technique works by either removing edges or contracts them showing a better way of computing chromatic polynomial [27].

The chromatic polynomial, when assessed at λ, represents the quantity of different ways to fully color the graph utilizing colors. The emergence of the genus concept can be traced back to the four-color conjecture, which paved the way for the definition of the Tutte polynomial. This polynomial acts as a generalization of the chromatic polynomial, expanding it to two variables [28]. The chromatic polynomial associated with a graph G is a polynomial function that reflects the number of distinct ways to color the graph using a certain number of colors [29].

The inherently simple and general nature of the deletion-contraction method [30] carried over well to applying it to polynomials with path-based limitations to get at formulations of P_3-chromatic polynomials. Such an advancement enabled analysis of the relationship of path constraints, especially those having the length equal to 2, to coloring properties. Thus, this approach was sustainable for the systematic examination of the P_3-chromatic polynomial that deals colorings subjected to some path constraints. The P_3-chromatic polynomial is used to find the quantity of valid coloring arrangements for a graph under P_3-coloring rules. The P_3-chromatic polynomial represents a significant development in graph theory, enhancing the traditional chromatic polynomial by incorporating constraints associated with paths of length 2 (i.e., up to three vertices). This extension enables researchers to address problems where the structure of local paths plays a critical role. Unlike the classical chromatic polynomial, which focuses solely on assigning colors to nodes/vertices so that neighboring ones do not share the same color, the P_3-chromatic polynomial introduces additional conditions. These ensure that every induced subgraph resembling a P_3 path adheres to proper coloring rules. While this increases the complexity, it can surprisingly simplify the resolution of certain intricate graph-related challenges. As graphs grow in size and complexity, many problems become difficult to manage. The P_3-chromatic polynomial offers a structured mathematical approach that encapsulates both the local and global characteristics of graphs.

Additionally, it uncovers symmetries and patterns within graph colorings, facilitating the analysis and decomposition of graphs into smaller, more manageable components. The P_3-chromatic polynomial is not limited to theoretical studies in graph analysis; it also has practical applications in various real-world scenarios that involve path-based constraints. This has practical applications in scheduling scenarios where tasks or events, represented by vertices, need to be allocated time slots or resources without overlaps. For instance, in creating a school timetable, P_3-coloring can effectively model constraints to ensure that classes sharing teachers or rooms do not occur at the same time. In communication networks, where nodes correspond to devices and edges represent direct communication links, P_3-chromatic polynomial assist in designing systems to minimize interference. For example, they can be used to allocate frequencies to communication towers, ensuring that towers located close to each other operate without overlapping frequencies. In road networks, intersections are represented as vertices and roads as edges. The P_3-chromatic polynomial can be utilized to optimize traffic light sequences, facilitating smoother traffic flow by minimizing conflicting movements. The P_3-chromatic polynomial provides a means to enumerate valid colorings that adhere to P_3-constraints when examining specific graph families, such as cycles, ladders, or bipartite graphs. This tool enhances the understanding and characterization of these graphs. The P_3-chromatic polynomial was thus derived as an extension of the traditional chromatic polynomial which is supposed to capture certain path based coloring restrictions. As the number of colorings restricted by conditions for paths of length 2 (i.e., upto 3 vertices), P_3-chromatic polynomial defines and effectively helps evaluate colorings in the graphs where such paths are more limited. Thus, this polynomial responds to coloring questions of classes of graphs for which individual chromatic polynomials fail. For example, P_3-chromatic polynomial has been investigated in the case of cycle graphs, wheel graphs and grid graphs whereby path based constraints are a primary determinant of colouring. Realizations of the P_3-chromatic polynomial in network theories and other ordered structures including the communication networks or biological systems reveal the importance of the polynomial in cases where path lengths are significantly determinative of the possible colorings. Given that the P_3-chromatic polynomial is difficult to compute for some graphs, let alone for large graphs, the task is challenging. Subsequent researches have proposed methods which are used to enhance P_3-chromatic polynomial in an efficient manner, due to the added computational difficulty of large or complex structures within a graph. The work with such methods to this date remains in progress, as graph theorists work on the optimization strategies for given polynomial computations together with studying the behavior of the polynomial relating to different classes of graphs. More fundamental questions remain, concerning the overall behavior of the P_3-chromatic polynomial, including highly connected graphs and even higher dimensional graphs. The P_3-chromatic polynomial proves useful in modeling and addressing such challenges by accounting for both local and global properties of the graph. In multiplayer games or sports tournaments, the P_3-chromatic polynomial aids in organizing schedules to prevent conflicts, ensuring that specific teams or players (represented as vertices) do not compete against each other under given constraints. The P_3-chromatic polynomial builds upon the traditional chromatic polynomial, introducing an additional level of combinatorial complexity and broader applicability. The P_3-chromatic polynomial proves especially valuable in addressing problems involving paths of specific lengths. It is particularly applicable in the study of graphs where P_3-induced subgraphs significantly influence properties such as connectivity or independence. When examining specific graph families like cycles, ladders, or bipartite graphs, the P_3-chromatic polynomial provides a method to enumerate valid colorings that adhere to P_3-constraints, offering deeper insights into their structural characteristics. Through the study of P_3-colorings, graph theorists can gain valuable insights into the decomposition of graphs into smaller subgraphs, which helps in tackling larger graph-related problems or proving theoretical results. The P_3-chromatic polynomial also has the potential to inspire the creation of new graph invariants or polynomials, broadening the understanding of chromatic properties for particular applications and paving the way for further advancements in graph theory.

2 Main Results

The term P_k-coloring was first used by Sopena when studying homomorphisms with cycles. A P_k-coloring of a graph G is a proper vertex coloring when there exists a integer $k \geq 3$, so that no path with exactly three vertices shares any two nodes having the identical color. They included colorings that were important for usage for example where one was trying to develop a schedule. year later in 2003, Truszczynski presented a completely other conceptualization of the chromatic polynomial which is called the P_k-Chromatic polynomial with a notation $\chi_k(G, \lambda)$ defines the number of P_k-coloring using λ color.

In other words, the idea of the P_3-chromatic polynomial has its root deep in more than half a century of significant study on colorings, chromatic polynomials, and P_k-colorings. Exploration of the P_3-chromatic polynomial roots and its appli-cations is the present day research area of graph theory owing to all the continuous growth and development in graph theory for more than two and a half centuries.

Definition 2.1. Coloring of vertices is a process in which we color vertices or nodes of a graph such that no two neighboring nodes/junctions share the uniform color.

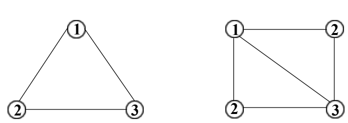
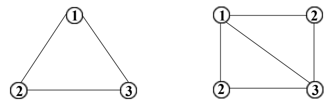


Figure 1: Vertex Coloring

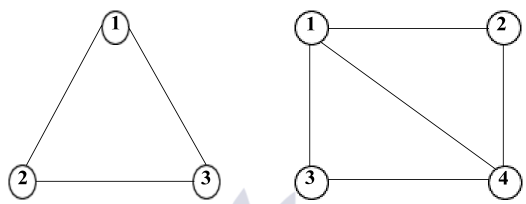
Definition 2.2. The notation $\chi(G)$ represents the chromatic number of a graph G, which is defined as the least quantity of colors necessary to color the graph's nodes without allowing any two neighboring nodes/junctions to be of the same color.



Chromatic Number = 3
Chromatic Number = 3

Figure 2: Chromatic Number

Definition 2.3. A simple graph is referred to as P_3-colorable if it can be colored in a manner that ensures all vertices in any P_3 path are assigned distinct colors. This technique is referred to as P_3-coloring of the graph.
Definition 2.4. The least quantity of colors which is mandatory to color a graph with P_3-coloring is called P_3-chromatic number.



P_3- Chromatic Number = 3
P_3- Chromatic Number = 4

Figure 3: P_3-Chromatic Number

Definition 2.5. Represented as $P(G, t)$, the chromatic polynomial of a graph serves as a function that indicates how many proper colorings can be achieved for a graph G with t available colors.

Chromatic polynomial of certain graphs:
Path graph $P_n = t(t-1)$
Cycle graph $C_n = (t-1)^n + (-1)^n(t-1)$
Complete graph $K_n = t(t-1)(t-2)\ldots,(t-(n-1))$
Definition 2.6. Square of a graph G (denoted by G^2) is a graph in which two vertices are adjacent iff they have distance of at most 2 in G. That means G^2 is created from G by adding an edge connecting vertices whose distance is at most 2.

G G^2

Figure 4: G^2 of graph

Definition 2.7. Let G be a graph and e_1 be an edge of G. Then

$$\boldsymbol{P(G), c = P(G - e1, c) - P\left(\frac{G}{e1}, c\right)},$$

Deletion: $G - e_1$ represents the line elimination e_1 from graph G.
Contraction: The notation $\frac{G}{e1}$ denotes the contraction of the edge $e1$ in the graph G, where this edge previously connected two vertices that have now been unified.

For every positive integer *q*, *P* (G, *c*) represents the number of *c*-colorings of the graph $G - e_1$ such that vertex *v* is colored differently from vertex *w*. On the other hand, $\boldsymbol{P}\left(\frac{\boldsymbol{G}}{\boldsymbol{e1}}, \boldsymbol{c}\right)$, indicates the number of *c*-colorings in which both vertices *v* and *w* 1
are assigned the same color.

Theorem 2.1. *Define C_n as the cycle graph, with the stipulation that $n \not\models 5$. This is valid for all integers n that are atleast 3 [12].*
$$x_3(C_n)\begin{cases} 3, n \quad \equiv 0(mod\ 3)\ ; \\ \ 4, n \cong 2(mod\ 3)\ . \end{cases}$$
Theorem 2.2. *Consider the ladder graph denoted as L_{2n}, which consists of 2n vertices. It follows that for every $n \geq 3$, the chromatic number $x_3(L_n)$ equals 4.*
Theorem 2.3. *Consider the path graph denoted as P_n. It follows that for every $n \geq 3$, the chromatic number $x_3(P_n)$.*
Corollary 2.4. *Analyzing a complete graph K_n with n vertices leads to the con-clusion that $x_3(K_n)$ = n for every n that is at least 3;*
Corollary 2.5. *A wheel graph W_n, which has n nodes, satisfies the condition that $x_3(W_n)$ = n for every n that is 4 or higher.*
Corollary 2.6. *Let us analyze a star graph labeled S_n containing n nodes. It is evident that $x_3(S_n)$ = n holds true for all values of n that are 3 or greater.*

2.1 Properties of P_3-chromatic polynomial

The P_3-chromatic polynomial is defined by its recursive properties. It can be com-puted through deletion-contraction strategies, similar to the classical chromatic polynomial, but with supplementary terms that consider conditions specific to P_3. This approach allows for systematic calculations, even in the case of complex graphs. Furthermore, the P_3-chromatic polynomial serves as a representation of the structural features of the graph. The degree of this polynomial is aligned with the total number of vertices in the graph, and the coefficients reveal critical in-formation about the graph's connectivity and the distribution of its subgraphs. It remains unchanged under graph isomorphism, which implies that isomorphic graphs have the same P_3-chromatic polynomial. This feature makes it a significant resource for identifying non-isomorphic graphs.
In summary, the P_3-chromatic polynomial serves as a vital extension of the chromatic polynomial, offering richer insight into graph colorings that are determined by path-related constraints, thereby enhancing the analytical depth of graph theory.
This algorithmic framework supports the computation of the P_3-chromatic polynomial for a broad spectrum of graphs. Its versatility enables it to manage complex graph architectures, making it an important asset for theoretical studies and practical applications in the field of graph theory.
Definition 2.8. The P_3-chromatic polynomial of a graph G is a polynomial function that defines in how many ways we can color a graph with *x* colors so that the graph is P_3-colorable graph. It is denoted by $P_{\chi 3}(G, x)$.

2.2 Path Graph

Theorem 2.7. *Let P_n be a path graph , then,* $P_{x3}(P_{k+1}, 1 = j(j-1)(j-2)^{k-1}$
Proof. **Base Step:** For n = 3

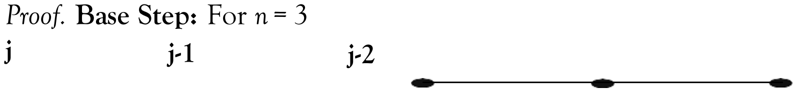


Figure 5: Path Graph P_3 $\boldsymbol{P_{x3}(P_3, j) = j(j-1)(j-2)}$

In P_3-coloring, the first vertex has j choices, second vertex has $j-1$ choices because it must be different from the first vertex, the third vertex has $j-2$ choices because in P_3-coloring we cannot assign the colors of two preceeding vertices to the particular vertex. So, the P_3-chromatic polynomial of this graph is $j(j-1)(j-2)$.

Inductive Hypothesis:

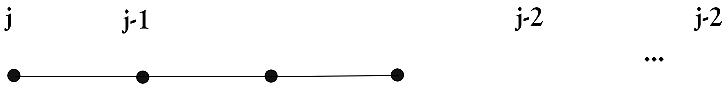


Figure 6: Path Graph P_k

Assume that this result is true for $n = k$. So, the P_3-chromatic polynomial becomes,

$$P_{x3}(P_k, j)_3 = j(j-1)(j-2)^{k-2}, k \geq 3$$

Inductive Step:
Now consider the path graph P_{k+1} with $k+1$ vertices,

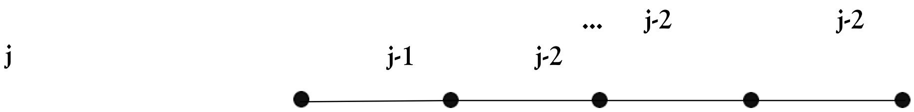


Figure 7: Path Graph P_{k+1} $P_{\chi 3}(P_{k+1}, j) = P_k(j).[j-(k, k-1)choices], \ k \geq 3$

$= j(j-1)(j-2)^{k-2}(j-2)$

$= j(j-1)(j-2)^{k-2+1}$

$= j(j-1)(j-2)^{k-1}$

□

Example:

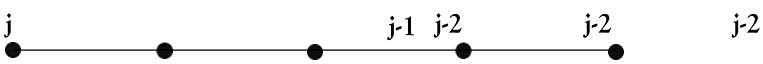


Figure 8: Path Graph

j j-1 j-2 j-2 j-2 j-2

Figure 9: Path Graph

Here we are applying P3-coloring for path graphs having 5 and 6 vertices as shown in Figure 8 and Figure 9 respectively. To find the P3-chromatic polynomial we have j choices of colors for the first vertex, j − 1 possible ways to color the second vertex because we cannot assign it the color of first vertex, j − 2 choices for the third vertex because in P3-coloring we can color in such a manner that ensures all vertices in any P3 path are assigned distinct colors. For the fourth vertex we cannot assign the color of preceeding two vertices, so it has also j −2 choices. Same case with the fifth and sixth vertices in above path graphs. So, the P3-chromatic polynomial of above graphs are j(j − 1)(j − 2)3 and j(j − 1)(j − 2)4 respectively.

Theorem 2.8. *The P3-chromatic polynomial of graph G is same as chromatic polynomial of G2*
Proof. Let λ be the P3-coloring of graph G, then the color of vertices on every P3 path are distinct. In other words, all the vertices that are at the distance 2 have different colors. Also the colors of all the vertices are distinct. This shows that λ coloring of G is also a coloring on G2.

Conversely, suppose that α be a coloring of G2. The edges of G2 are of two types, the one that belongs to E(G) and the one that do not belongs to E(G) also the edges between the vertices that are at the distance 2 in G. Since α is coloring on G2 then the colors of all the vertices that are at the distance 2 in G are distinct. So, if P3 is any path in G. Then the colors of its nodes/junction are distinct under α. So, α is P3-coloring.
P_x3 (G,x)=P(G^2,x)
The following result is an analogues of the deletion and contraction theorem for chromatic polynomial of a graph.

Corollary 2.9. *(Deletion-Contraction version of P_3 coloring) For a graph G, the P_3-chromatic polynomial of G is:*

$P_{x3}(G,t) = P(G^2 - e_1, t) - P\left(\frac{G^2}{e_1}, t\right)$, *where e_1 is any arbitrary edge of G^2. Proof.* The proof is clear from the Theorem 2.2 and from the deletion contraction definition 2.7.

□

Corollary 2.10. *Consider G as a graph and let $G^2, G^2, \dots, G^2$ be connected components of G^2. Then,*

$$P_{x3}(G,u) = P(G_1^2,u).P(G_2^2,u).....P(G_k^2,u)$$

Proof

$$P(G^2,u) = P(G_1^2,u).P(G_2^2,u).....P(G_k^2,u)$$

This proof is clear from the Theorem 2.2

□

Corollary 2.11. *The P_3-chromatic polynomial of empty graph is*
$P_{x3}(G,w) = w^n$
Proof. This proof is evident from the Theorem 2.2

□

Corollary 2.12. *Let G represent a graph, then the degree of its P_3-chromatic polynomial $P_{\chi 3}(G,x)$ corresponds to the total number of nodes in the graph G.*

Corollary 2.13. *Consider a graph G with P_3-chromatic polynomial $P_{\chi 3}(G,x)$. Then the following are true:*

1. The leading coefficient of $P_{\chi 3}(G,t)$ of any graph is 1.
2. The absolute value of coefficient of the t^{n-1} term in $P_\chi(G,t)$ is the number of lines/edges.
3. The first coefficient of $P_{\chi 3}(G,t)$ is positive, and all terms alternate in sign.
4. All coefficients are integers.

Corollary 2.14. *Let K_n be a complete graph then its P_3-chromatic polynomial will be,*

$$P_{x3}(K_n, t) = t(t-1)\ldots.,(t-n+1)$$

Corollary 2.15. *Consider a graph G characterized by its P_3-chromatic polynomial, represented as $P_{x3}(G, q)$. Then G^2 is connected iff the coefficient of q in$P_{x3}(G, q)$ is nonzero.*

Remark 2.1.

- Consider a graph denoted as G and let $P(x)$ represent its chromatic polyno-mial. The chromatic number of G is defined as the smallest positive integer k such that $P(k)$ is greater than zero.
- As $P_{x3}(G, x) = P_x(G^2, x)$, the smallest positive integer k satisfying $P_{x3}(k) > 0$ indicates the P_3-chromatic number of G.

Remark 2.2. Let G be a graph of diameter 2. Then,

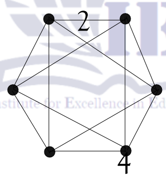
$P_{\chi}3\,(D_2, x) = P(K_n, x)$

Theorem 2.16. *For $n = 3k$, the $P_{\chi}3\,(C_n, m) = m(m-1)(m-2)q(m)$ where $q(m)$ is a polynomial of $3(k-1)$ degree such that $P_{\chi}3\,(C_n, 3) > 0$*

Proof. By Theorem 2.1, the P_3-chromatic number of cycle graph on $n = 3k$, where k is any positive integer, is 3. So, atleast 3 colors are needed for P_3 coloring of cycle graph and by Remark 2.1 we have $P_{\chi}3\,(C_n, 3) > 0$. Therefore, as a factor, the term $m(m-1)(m-2)$ remains the same for the P_3-chromatic polynomial of cycle graphs of vertices $n = 3k$. Moreover, from Corollary 2.12, it is clear that the degree of these types of graph is $3k$. So, from these two arguments it is evident that $q(m)$ has $3(k-1)$ degree.

□

1

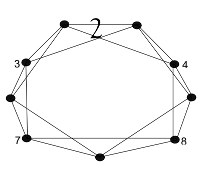


3

5 6

Figure 10: G^2 of C_6

1



5 6

9

Figure 11: G^2 of C_9

Examples:

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$P_{\chi_3}(C_6, m) = m(m-1)(m-2)(m^3 - 9m^2 + 29m - 32)$

$P_{\chi_3}(C_9, m) = m(m-1)(m-2)(m^6-15m^5+97m^4-351m^3+760m^2-939m+514)$

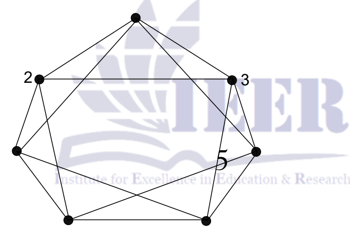
In Figure 10 and Figure 11, we have considered the cycle graphs of 6 and 9 vertices. As we know that P_3-coloring of graph G is same as chromatic polynomial of G^2. So, by joining the vertices which are at the distance of two we make a graph G^2, now finding the chromatic polynomial of G^2 that is, we cannot give same color to the two adjacent vertices. From Theorem 2.1, the chromatic number of cycle graph having $3k$ vertices is 3, so the term $m(m-1)(m-2)$ remains the same for all cycle graphs of $3k$ vertices and the remaining polynomial vary according to the number of vertices in graph. Moreover, from Corollary 2.12, the degree of its P_3-chromatic polynomial is 6 and 9 respectively.

Theorem 2.17. *For $n \neq 3k$, the polynomial $P_{\chi_3}(C_n, u)$ can be expressed as $P_{\chi_3}(C_n, u) = u(u-1)(u-2)(u-3)r(u)$, where $r(u)$ is a polynomial of degree $n-4$ and satisfies the condition $P_{\chi_3}(C_n, 4) > 0$.*

Proof. By Theorem 2.1 the P_3-chromatic number of cycle graph on $n \neq 3k$, where k is any positive integer, is 4. So, atleast 4 colors are needed for P_3-coloring of cycle graph and by Remark 2.1 we have $P_{\chi_3}(C_n, 4) > 0$. Thus, the factor $u(u-1)(u-2)(u-3)$ remains unchanged in the P_3-chromatic polynomial of cycle graphs for cases where the number of vertices n does not equal $3k$. Moreover, from Corollary 2.12, it is clear that the degree of these types of graph is n. So, from these two arguments it is evident that $r(u)$ has $n-4$ degree. □

Examples:

1



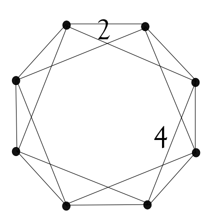
4

6 7

Figure 12: G^2 of C_7

$P_{\chi_3}(C_7, u) = u(u-1)(u-2)(u-3)(u^3 - 8u^2 + 25u - 29)$ $P_{\chi_3}(C_8, u) = u(u-$

1



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5

7

Figure 13: G^2 of C_8 1)$(u-2)(u-3)(u^4 - 10u^3 + 41u^2 - 84u + 71)$

In case of vertices $n \neq 3k$, let us consider the graphs of vertices 7 and 8 as depicted in Figure 12 and Figure 13 correspondingly. As established $P_{\chi_3}(G, x) = P(G^2, x)$, we make a graph G^2 finding the chromatic polynomial of G^2

gives us the P_3-chromatic polynomial of graph G. From Theorem 2.1, for cycle graphs with $n \neq 3k$ vertices, the chromatic number is 4, ensuring that the factor $u(u-1)(u-2)(u-3)$ is unchanged, with variations in the rest of the polynomial based on the graph's vertex count. Moreover, from Corollary 2.12, the degree of its P_3-chromatic poly-nomial is 7 and 8 respectively.

Theorem 2.18. ***For*** $2n$ ***vertices the*** $P_{\chi 3}(L_{2n}, s) = s(s-1)(s-2)(s-3)(s^2-7s+13)^k$ ***where*** $k = n-2$ ***such that*** $P_{\chi 3}(L_{2n}, 4) > 0$

Proof. By Theorem 2.2, the P_3-chromatic number of ladder graph on n vertices, is 4. So, atleast 4 colors are needed for P_3-coloring of ladder graph and by Remark 2.1 we have $P_{\chi 3}(L_{2n}, 4) > 0$. Therefore, as a factor, the tern□$s(s-1)(s-2)(s-3)(s^2-7s+13)$ remains the same for the P_3-chromatic polynomial of ladder graphs on nodes/junctions $2n$. Moreover, from Corollary 2.12, it is clear that the degree of these types of graph is $2n$. By fixing $n-2$ as the degree of $(s^2-7s+13)$ a solving the powers of s it becomes s^{2n}. So, from these two arguments it is evident that the degree of ladder graph is $2n$.

Example:

Figure 14: G^2 **of** L_6 $P_\chi(L_6, s) = s(s-1)(s-2)(s-3)(s^2-7s+13)$

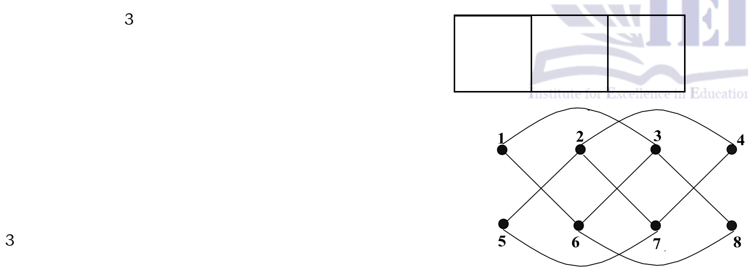

Figure 15: G^2 **of** L_8 $P_\chi(L_8, s) = s(s-1)(s-2)(s-3)(s^2-7s+13)^2$

Here we have taken ladder graphs with 6 and 8 vertices as illustrated in Figure 14 and Figure 15 respectively, the chromatic number is 4, ensuring that the factor $s(s-1)(s-2)(s-3)$ is unchanged, while the remainder of the polynomial varies according to the vertex count. Moreover, from Corollary 2.12, it is evident that the degree of its P_3-chromatic polynomial is 6 and 8 respectively.

Theorem 2.19. ***Let*** S_n ***be defined as a star graph containing n nodes. It can be stated that*** $P_{\chi 3}(S_n, h) = P(K_n, h)$ ***and the*** P_3***-chromatic number*** $\chi_3(S_n) = n$
$\forall n \geq 3$

Proof. Suppose S_n is a star graph having n nodes/vertices ($n \geq 3$). As G^2 of this graph S_n will be isomorphic to K_n, so from Theorem 2.2

chromatic polynomial of G^2 = P_3-chromatic polynomial of S_n= chromatic polyno-mial of K_n

Furthermore, as indicated by Corollary 2.6, the P_3-chromatic number for a star graph is n, leading to the conclusion that $\chi_3(S_n) = n$. □

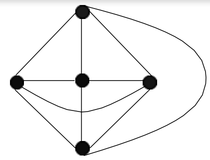

Figure 16: G^2 of S_5

Example:

$P_{\chi 3}(S_5, h) = h(h-1)(h-2)(h-3)(h-4)$

let us consider a star graph having 5 vertices(see Figure 16). By connecting the vertices which are at the distance two, makes a complete graph K_5 and in complete graph all the vertices are at the distance two so we have to assign different colors to all the nodes/junction of complete graph. Then, the P_3-chromatic polynomial of star graph S_5 becomes $h(h-1)(h-2)(h-3)(h-4)$ and the P_3-chromatic number which is the least number of colors needed to achieve a valid P_3-coloring is 5.

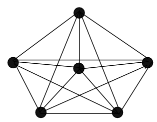

Figure 17: G^2 of S_6 $P_{\chi 3}(S_6, h) = h(h-1)(h-2)(h-3)(h-4)(h-5)$

Considering another star graph S_6 containing 6 vertices as shown in Figure 17.

By joining the vertices which are at the distance two, makes it a complete graph K_6 and in complete graph all the vertices are at the distance two so we have to assign different colors to all the nodes/junctions of complete graph. Then, the P_3-chromatic polynomial of star graph S_6 becomes $h(h-1)(h-2)(h-3)(h-4)(h-5)$ and the P_3-chromatic number which is the least quantity of colors necessary to achieve a valid P_3-coloring is 6.

Theorem 2.20. ***Consider W_n as a wheel graph consisting of n nodes. It can be established that $P_{\chi 3}(W_n, r) = P(K_n, x)$, and the P_3-chromatic number $\chi_3(W_n) = n$***

$\forall n \geq 4$

Proof. Suppose W_n is a wheel graph having n nodes/vertices ($n \geq 4$). As G^2 of this graph S_n is isomorphic to K_n, so from Theorem 2.2

chromatic polynomial of G^2 = P_3-chromatic polynomial of W_n = chromatic poly-nomial of K_n

Furthermore, according to Corollary 2.5, the wheel graph has a P_3-chromatic num-ber of n, thus $\chi_3(S_n) = n$.

□

Example:

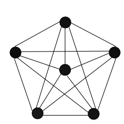

Figure 18: G^2 of W_6 $P_{\chi 3}(W_6, r) = r(r-1)(r-2)(r-3)(r-4)(r-5)$

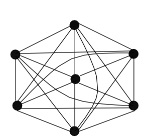

Figure 19: G^2 of W_7 $P_{\chi 3}(W_7, r) = r(r-1)(r-2)(r-3)(r-4)(r-5)(r-6)$

Here we have considered two wheel graphs having 6 (see Figure 18) and 7 (see Figure 19) vertices. By linking the vertices that are at the distance two, it makes a complete graph K_6 and K_7 correspondingly, and the complete graph is a graph of diameter 2 (i.e., all the vertices are at the distance 2). So, we have to assign different colors to all the vertices of the complete graph. The P_3-chromatic number of the wheel graphs W_6 and W_7 is 6 and 7 respectively.

3 Conclusion

In our study, we proposed and investigated in detail a new graph coloring scheme that we call P_3- coloring. Being a generalization of vertex coloring, P_3- coloring places path coloring constraints, extending the classical definition of graph theory. One of the most important contributions of this work was the fact that in any graph with a universal vertex (a vertex that is adjacent to all other vertices), the P_3- chromatic polynomial is the cardinality of the set of vertex. In addition, we proved that the P_3- chromatic number of a graph is always greater than or equal to its classical chromatic number, giving a new twist of complexity to classical coloring problems. Beyond the numerical values, the study explored the P_3- chromatic polynomial, which is an essential extension of the classical chromatic polynomial.

We used the fundamental theory of vertex coloring and P_3- coloring to compute these polynomials systematically in a few common graphs, such as: Path graphs and Cycle graphs, ladder graphs, and star graphs and wheel graphs The calculation of these polynomials contributed to the theory of algebraic graph theory in general, providing an important understanding of the structure of such kinds under the condition of P_3- coloring. Due to practical issues, which can only be addressed successfully in terms of coloring graphs with the color set of P_3- coloring, the introduction of this technique was understandable, which conditions the need to conduct a series of subsequent studies. This paper ends with some open problems and future research directions, in a bid to further develop the use of the application of P_3- coloring and its polynomials in analyzing complex networks and designing combinatorics.

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