A COMPREHENSIVE NOTE ON WEAK-COMMUTATIVE AG-GROUPOIDS

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Abstract

*A magma satisfying the left invertive law, $ab \cdot c = cb \cdot a$ is known as an AG-groupoid or left-almost semi-group. The concept of bi-commutative groupoid arising in [*N. Bobeica et al. (2010)*] is extended to introduce a new subclass of AG-groupoid as weak-commutative AG-groupoid. Latest computational techniques of Mace-4 and GAP are used for generating various non-associative examples to prove existence of this class. Enumeration of this subclass up to order 6 is also provided. Various relations of this class with other known algebraic structures are established. Further, the newly introduced class of weak-commutative AG-groupoids is decomposed by defining some congruences.*

INTRODUCTION

Bi-commutative groupoids are defined and investigate in (N. Bobeica et al. (2010)). The concept of these groupoids is considered here to introduce a new subclass of AG-groupoid. AG-groupoids is in general a non-associative non- commutative structure lying midway through the groupoid and the commutative semigroup. AG-groupoid, is a magma that satisfies the left invertive law, $(ab)c = (cb)a$. Every AG-groupoid is medial, i.e. it satisfies the medial law, $(ab) \cdot (cd) = (ac) \cdot (bd)$. We use juxtaposition and the notation " $\cdot$ " to avoid frequent use of parenthesis, e.g. $((a \cdot b) \cdot c)d$ will be the same as $((ab)c)d$.

AG-groupoids have a variety of applications in flock theory, algebra, finite mathematics, topology and geometry (N. Bobeica et al. (2010), Kazim & Naseerudin,(1977), Shah,(2013), Karaaslan et al.,(2016), Amanullah, (2014)). Various subclasses of AG-groupoids have been introduced and investigated by various others researchers [M. Iqbal et al.,(2018), I. Ahmad et al., (2020), Ahmad et al., (2020), Ahmad et al., (2016)] and recently some more subclasses have been introduced in (M. Rashad et al.,(2018), M. Shah et al.,(2012), M. Rashad et al.,(2013), I. Ahmad et al.,(2013), M. Rashad et al., (2014)). Ideals in some of these classes are defined and investigated (Amanullah et al., (2018), M. Rashad et al. (2018), M. Rashad et al. (2019)). In section 2, we define our new subclass of AG-groupoid, as weak-commutative AG-groupoid. We provide non- associative examples using the modern computational techniques to prove the existence of this class as has been for various other subclasses. In Section 3, we discuss some important properties of weak-commutative AG-groupoids for deeper exploration. The phenomenon of construction of some other algebraic structures from weak-commutative AG-groupoid and vice versa is included in Section 4 via a series of theorems:AG-groupoid and an LP-AG-groupoid can be constructed under some specific conditions from weak- commutative groupoid.

(i) An AG-groupoid S can be converted to a weak-commutative AG-groupoid for some $\alpha, \beta \in End(S)$.

(ii) A commutative semigroup can be constructed from a weak-commutative AG-groupoid.

The enumeration and classification of weak-commutative AG-groupoids up to order 6 is included in Section 5. The relations, listed below are congruence on a weak-commutative AG-groupoid as proved in Section 6.

(i) A relation η defined on a weak-commutative AG-groupoid S as $a\eta b \Leftrightarrow xa = yb$ for $a, b \in S$ and for some $x, y \in S$ is congruence.

(ii) A relation ρ defined on a weak-commutative AG-groupoid S as $a\rho b \Leftrightarrow ea = eb$ for every $e \in E(S)$, where $E(S)$ is the set of idempotent elements of S, is congruence.

2. PRELIMINARIES

In the following we list some definitions arising in (F. Karaaslan et al.(2016), I. Ahmad et al.,2016) and that will frequently be used throughout this article. An AG-groupoid S is called —

(n) — is called AG*-groupoid if it satisfies the identity $ab \cdot c = b \cdot ac$,

(a) — paramedial AG-groupoid if it satisfies the identity $ab \cdot cd = db \cdot ca$

(b) — called T^1-AG-groupoid if $\forall a, b, c, d \in S$ it satisfies the identity $ab = cd \longrightarrow ba = dc$.

(c) — called T^4-AG-groupoid if it holds the identity $ab = cd \longrightarrow ad = cb, \forall a, b, c, d \in S$.

(d) — called T^4-AG-groupoid if $ab = cd$ then $da = bc, \forall a, b, c, d \in S$.

(e) — called T^4-AG-groupoid if it is both T^4 and T^4.

(f) — T^3-AG-groupoid if it satisfies the identity $ba = ca \longrightarrow ab = ac$

(g) — is called T^3-AG-groupoid if it satisfies the identity $ab = ac \longrightarrow ba = ca$

(h) — T^3-AG-groupoid if it both T^3 and T^3-AG-groupoid.

(i) — is called BC-AG-groupoid if it is both LC-AG-groupoid (i.e satisfies the identity, $ab \cdot c = ba \cdot c$) and RC -AG-groupoid (i.e satisfies the identity, $a \cdot bc = a \cdot cb$) $\forall a, b, c, d \in S$.

(j) — is called permutable AG-groupoid if it is right permutable (i.e satisfies the identity, $ab \cdot c$) as well as left permutable AG-groupoid (i.e satisfies the identity, $a \cdot bc = b \cdot ca$) $\forall a, b, c \in S$.

(k) — is called an AG-band if it satisfies the identity $aa = a$,

(l) — is called a semigroup if it satisfies the associative law, $ab \cdot c = a \cdot bc$,

(m) is called commutative if $ab = ba$, $\forall a, b \in S$.

(o) — is called right nuclear square AG-groupoid if it satisfies the identity $ab \cdot c^2 =$

$a \cdot bc^2$,

(p) — is called self-dual AG-groupoid if it satisfies the identity $a \cdot bc = c \cdot ba$,

(q) — is called Bol*-AG-groupoid

(r) — is called AG**-groupoid

(s) A relation $\leq$ is a partial order on S, if it satisfies the following conditions;

(i) $\leq$ is reflexive that is $a \leq a \quad \forall a \in S$.

(ii) $\leq$ is antisymmetric that is $a \leq b$ and $b \leq a \Leftrightarrow a = b \quad \forall a, b \in S$.

(iii) $\leq$ is transitive that is $a \leq b,\ b \leq c \rightarrow b \leq c \quad \forall a, b, c \in S$.

An AG-groupoid S is called inverse AG-groupoid if for every $a \in S$ there exists $x \in S$ such that $(ax)a = a$ and $(xa)x = x$. Here by x we shall mean an inverse of a (Aziz-ul-Hakim et al.,(2016). S is called completely inverse AG-groupoid if $ax = xa$. By $V(a)$ we shall mean the set of all inverses of a (M. Rashad et al., (2020)).

3. Existence of Weak-commutative AG-groupoids

Definition 3.1. *An AG-groupoid S is called a* ***weak-commutative AG-groupoid*** *(WC-AG-groupoid) if for all a, b, c, d* $\in S$,

$$ab \cdot cd = dc \cdot ba \tag{3.1}$$

Example 1. *Let* $S = \{1, 2, 3, 4\}$. $(S, *)$ *and* $(S, \cdot)$ *are non-associative WC-AG-groupoids of order* 4.

$*$	1	2	3	4
1	1	2	2	2
2	2	1	1	1
3	2	1	1	1
4	3	1	1	1

$\cdot$	1	2	3	4
1	1	1	3	3
2	1	1	4	4
3	3	3	1	1
4	3	3	1	1

4. Characterization of weak commutative Ag-groupoids

In this section we characterize WC-AG-groupoid. We prove that;

1. Every WC-AG-groupoid is a paramedial AG-groupoid and is a semigroup if it is an AG-band. Furthermore, a WC-AG-groupoid is right nuclear square if it is self-dual.
2. Each of the following is a WC-AG-groupoid. (i). T^1-AG-groupoid (ii). T^4-AG-groupoid (iii). BC-AG-groupoid (iv). RP-AG-groupoid (v). LP-AG-groupoid (vi). AG*-groupoid.

In the following counterexample it is shown that neither self-dual AG-groupoid nor weak-commutative AG- groupoid is right nuclear square AG-groupoid.

Example 2. *(i) Weak commutative AG-groupoid of order* 3, *which is not a right nuclear square AG-groupoid.*

(ii) Self-dual AG-groupoid of order 4, *which is not a right nuclear square AG-groupoid.*

(iii) WC-AG-groupoid of order 4, *which is not a commutative semigroup.*

·	1	2	3
1	1	1	1
2	1	1	1
3	1	2	2

(i)

*	1	2	3	4
1	1	3	4	2
2	4	2	1	3

(ii)

*	1	2	3	4
1	1	2	2	2
2	2	1	1	1

(iii)

However, in the following result we prove that the combination of these two classes gives us a right nuclear square AG-groupoid. We start with the following theorem.

Theorem 1. *Let* S *be a WC-AG-groupoid. Then each of the following hold.*

(i) S *is paramedial AG-groupoid;*

(ii) S *is commutative semigroup if it is AG-band;*

(iii) S *is right nuclear square if it is self-dual.*

Proof. Let S be a WC- AG-groupoid, and $a, b, c, d \in S$. Then

(i) By medial law and weak-commutativity, we have

$$ab \cdot cd = dc \cdot ba = db \cdot ca$$

Thus $ab \cdot cd = db \cdot ca$. Hence S is paramedial AG-groupoid.

(ii) Let S be a weak commutative AG-groupoid with AG-band property, and $a, b \in S$. Then, by AG-band property, weak commutativity, we have

$$ab = aa \cdot bb = bb \cdot aa = ba.$$

Thus $ab = ba$. Hence S is commutative, and since it is easy to show that commutativity implies associativity in AG-groupoid, so S is commutative semigroup.

(iii) Let S be a self-dual weak-commutative AG-groupoid, and $a, b, c \in S$. Then by self-duality and weak-commutativity we have

$$a \cdot bc^2 = c^2 \cdot ba = ab \cdot c^2.$$

Therefore, $a \cdot bc^2 = ab \cdot c^2$. Hence S is right nuclear square AG-groupoid.

Hence the theorem is proved.■

The following counterexample shows that every T^3-AG-groupoid is not a weak-commutative AG-groupoid.

Example 3. *T^3-AG-groupoid of order 4, which is not a weak-commutative AG-groupoid.*

*	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

Theorem 2. *Each of the following is a subclass of WC-AG-groupoid.*

(i) T^1-*AG-groupoid*

(ii) T^4-*AG-groupoid*

(iii) BC-AG-groupoid

(iv) RP-AG-groupoid

(v) LP-AG-groupoid

(vi) AG-groupoid.*

Proof. We proceed as follows.

(i) Let S be a T^1-AG-groupoid, and $a, b, c, d \in S$. Then by the definition of T^1-AG-groupoid and medial law, we get

$$ab \cdot cd = ac \cdot bd \quad \rightarrow \quad cd \cdot ab = bd \cdot ac \rightarrow ca \cdot db = bd \cdot ac \rightarrow db \cdot ca = ac \cdot bd$$

Thus by medial law, $dc \cdot ba = ab \cdot cd$. Hence S is weak-commutative AG-groupoid.

(ii) Let S be a T^4-AG-groupoid, and $a, b, c, d \in S$. Then by definition of T^4-AG-groupoid and medial law we have

$$ab \cdot cd = ac \cdot bd \quad \rightarrow \quad ab \cdot bd = ac \cdot cd$$

$$\rightarrow cd \cdot ab = bd \cdot ac \quad \rightarrow \quad ca \cdot db = ba \cdot dc$$

$$\rightarrow dc \cdot ca = db \cdot ba \quad \rightarrow \quad dc \cdot ba = db \cdot ca$$

$$\rightarrow dc \cdot ba = ab \cdot cd.$$

Hence S is weak-commutative AG-groupoid.

(iii) Let S be a BC-AG-groupoid, and $a, b, c, d \in S$. Then using the properties of BC-AG-groupoid and the medial law we have,

$$ab \cdot cd = ba \cdot cd = ba \cdot dc = bd \cdot ac = db \cdot ac = db \cdot ca = dc \cdot ba$$

Thus $ab \cdot cd = dc \cdot ab.$ Hence S is weak-commutative AG-groupoid.

(iv) Let S be a right permutable AG-groupoid, and $a, b, c, d \in S$. To prove that S is weak-commutative AG-groupoid, we use the left invertive law, the medial law and the definition of an RP-AG-groupoid, we get

$$ab \cdot cd = (cd \cdot b)a = (cb \cdot d)a = (db \cdot c)a = ac \cdot db$$

$$= ad \cdot cb = (a \cdot cb)d = (d \cdot cb)a = da \cdot cb = dc \cdot ab = (ab \cdot c)d$$

$$= (ac \cdot b)d = (bc \cdot a)d = (ba \cdot c)d = dc \cdot ba.$$

It means that $ab \cdot cd = dc \cdot ab.$ Hence S is weak-commutative AG-groupoid.

(v) Now for left permutable AG-groupoid, again using medial law and the definition of an LP-AG- groupoid we have

$$ab \cdot cd = ac \cdot bd = b(ac \cdot d) = b(dc \cdot a) = dc \cdot ba$$

Thus $ab \cdot cd = dc \cdot ab.$ Hence S is weak-commutative AG-groupoid.

(vi) Let S be an AG*-groupoid, and $a, b, c, d \in S$. Then, by definition of AG*-groupoid and repeated use of left invertive law, we have

$$ab \cdot cd = (cd \cdot b)a = (d \cdot cb)a = (a \cdot cb)d = (ca \cdot b)d = (ba \cdot c)d = dc \cdot ba$$

Equivalently, $ab \cdot cd = dc \cdot ab.$ Hence S is weak-commutative AG-groupoid.

Hence the theorem is proved. ■

Theorem 3. (I. Ahmad et al, (2013)) *(i). Every Bol*-AG-groupoid with left identity is a T^1-AG-groupoid.*

(ii). Every T^2-AG-groupoid is T^1-AG-groupoid.
*(iii). Every cancellative AG** is T^1-AG-groupoid.*

Using Theorem 2, the following corollary is obvious.

Corollary 1. *(i). Every Bol*-AG-groupoid with left identity is a WC-AG-groupoid.*

(ii). Every T^2-AG-groupoid is WC-AG-groupoid.
*(iii). Every cancellative AG**-groupoid is WC-AG-groupoid.*

5. Construction of Algebraic Structures

Construction of algebraic structure is always important for its development. The examples so achieved are sometimes not even possible through computers. Sometimes even open problems and conjunctures are often answered by constructing examples for them. A variety of constructions are available for forming quasigroup and loops and other algebraic structures. Sometimes construction can be implemented on a computer which then makes the job easier. For example, the construction of AG-groups from Abelian groups has been implemented in GAP. Several types of structure can be obtained from each other through some specific constructions. For example, this has been done for AG-groupoids (M. Rashad et al., (2012)).

We discuss some constructions of various algebraic structures from WC-AG-groupoid and vice versa via a series of theorems.

In Theorem 4 we prove that for any fixed element c of a weak commutative groupoid S, we define " $*$ " in such a way that we get an AG-groupoid while in Theorem 5 for a given $(S, *)$ AG-groupoid we prove that for some $\alpha, \beta \in End(S)$ we can define " $\cdot$ " on S, such that $(S, \cdot)$ is a weak-commutative groupoid.

Theorem 4. *Let $(S, \cdot)$ be a weak-commutative groupoid. Define $*$ on S as $x * y = x(cy)$, for some fixed $c \in S$. Then $(S, *)$ is an AG groupoid. In addition if $(S, \cdot)$ satisfies the identity $a(bc) = b(ac)$ then $(S, *)$ becomes an LP-AG-groupoid.*

Proof. Let $x, y, z \in S$. Then by definition of " $*$ ", weak commutativity and medial law, we have $(x * y) * z =$

$$(x(cy))(cz) = (zc)((cy)x) = (z(cy))(cx)$$

$$= (z * y) * x \rightarrow (x * y) * z = (z * y) * x.$$

Thus $(S, *)$ satisfies the left invertive law and hence is an AG-groupoid. Now let $(S, \cdot)$ satisfies the identity, $x(yz) = y(xz), \forall x, y, z \in S$. Then by repeated use of $x(yz) = y(xz)$ we have

$$x*(y*z) = x(c(y(cz))) = x(y(c(cz))) = y(x(c(cz)))$$

$$= y(c(x(cz))) = y*(x*z) \longrightarrow x*(y*z) = y*(x*z).$$

Hence $(S, *)$ is an LP-AG-groupoid. ■

Theorem 5. *Let $(S, *)$ be an AG-groupoid. Define "$\cdot$" as $x \cdot y = \alpha(x) * \beta(y)$, where $\alpha, \beta \in End(S)$. Then*

(1) $(S, \cdot)$ is weak commutative groupoid if $\alpha^2 = \beta^2$, $\alpha\beta = \beta\alpha$ and if any of the following holds.

(a) $(S, *)$ is an LP-AG-groupoid,
(b) $(S, *)$ is an AG^*-groupoid.

(2) $(S, \cdot)$ is weak commutative groupoid if α^2, β^2 are constant.

Proof. (1) Let $(S, *)$ be an AG-groupoid and

(a) Let $(S, *)$ be an LP-AG-groupoid and $a, b, c, d \in S$. Then by definition of "$\cdot$", we have

$$ab \cdot cd = \alpha(\alpha(a) * \beta(b)) * \beta(\alpha(c) * \beta(d))$$

$$= (\alpha^2(a) * \alpha\beta(b)) * (\beta\alpha(c) * \beta^2(d)) \qquad (5.1)$$

Similarly, again by definition of "$\cdot$", LP-AG-groupoid and left invertive law, we have

$$dc \cdot ba = \alpha(\alpha(d) * \beta(c)) * \beta(\alpha(b) * \beta(a))$$

$$= (\alpha^2(d) * \alpha\beta(c)) * (\beta\alpha(b) * \beta^2(a))$$

$$= [(\beta\alpha(b) * \beta^2(a)) * \alpha\beta(c)] * \alpha^2(d)$$

$$= [(\alpha\beta(c) * \beta^2(a)) * \beta\alpha(b)] * \alpha^2(d)]$$

$$= (\alpha^2(d) * \beta\alpha(b)) * (\alpha\beta(c) * \beta^2(a)) \qquad (5.2)$$

From Equations (5.1) and (5.2), $(S, \cdot)$ is weak commutative $\longrightarrow\longrightarrow$ $\alpha^2 = \beta^2$ and $\alpha\beta = \beta\alpha$.

(b) Let $(S, *)$ be an AG^*-groupoid and $a, b, c, d \in S$. Then by definition of "$\cdot$", medial law, left invertive law and definition of AG^*-groupoid we have

$$ab \cdot cd = \alpha(\alpha(a) * \beta(b)) * \beta(\alpha(c) * \beta(d))$$

$$= (\alpha^2(a) * \alpha\beta(b)) * (\beta\alpha(c) * \beta^2(d))$$

$$= [(\beta\alpha(c) * \beta^2(d)) * (\alpha\beta(b)] * \alpha^2(a)$$

$$= [(\alpha\beta(b) * \beta^2(d)) * \beta\alpha(c)] * \alpha^2(a)$$

$$= [\beta^2(d) * (\alpha\beta(b) * \beta\alpha(c))] * \alpha^2(a)$$

$$= [\alpha^2(a) * (\alpha\beta(b)) * \beta\alpha(c))] * \beta^2(d)$$

$$= (\alpha\beta(b) * \beta\alpha(c)) * (\alpha^2(a) * \beta^2(d)) \quad (5.3)$$

Similarly, applying the left invertive law, medial law and the definition of AG*-groupoid, we have

$$dc \cdot ba = \alpha(\alpha(d) * \beta(c)) * \beta(\alpha(b) * \beta(a))$$

$$= (\alpha^2(d) * \alpha\beta(c)) * (\beta\alpha(b) * \beta^2(a))$$

$$= [(\beta\alpha(b) * \beta^2(a)) * \alpha\beta(c)] * \alpha^2(d)$$

$$= \alpha\beta(c) * [(\beta\alpha(b)) * \beta^2(a)) * \alpha^2(d)]$$

$$= \alpha\beta(c) * [(\alpha^2(d)) * \beta^2(a)) * \beta\alpha(b)]$$

$$= [(\alpha^2(d)) * \beta^2(a)) * \alpha\beta(c)] * \beta\alpha(b)$$

$$= (\beta\alpha(b) * \alpha\alpha(c)) * (\alpha^2(d) * \beta^2(a)) \quad (5.4)$$

From Equations (5.3) and (5.4), $(S, \cdot)$ is weak commutative $\rightarrow\rightarrow$ $\alpha^2 = \beta^2$ and $\alpha\beta = \beta\alpha$.

(2) Let $(S, *)$ be an AG-groupoid and $a, b, c, d \in S$. Then by definition of "$\cdot$" and left invertive law, we have

$$ab \cdot cd = \alpha(\alpha(a) * \beta(b)) * \beta(\alpha(c) * \beta(d))$$

$$= (\alpha^2(a) * \alpha\beta(b)) * (\beta\alpha(c) * \beta^2(d))$$

$$= [(\beta\alpha(c) * \beta^2(d)) * \alpha\beta(b)] * \alpha^2(a) \quad (5.5)$$

Similarly, by definition of "$\cdot$" and left invertive law, we have

$$dc \cdot ba = \alpha(\alpha(d) * \beta(c)) * \beta(\alpha(b) * \beta(a))$$

$$= (\alpha^2(d) * \alpha\beta(c)) * (\beta\alpha(b) * \beta^2(a))$$

$$= [(\beta\alpha(b) * \beta^2(a)) * \alpha\beta(c)] * \alpha^2(d) \quad (5.6)$$

From Equations (5.5) and (5.6), $(S, \cdot)$ is weak commutative if

$\alpha^2(a) = \alpha^2(d)$, $\beta^2(d) = \beta^2(a)$ and $\beta\alpha(b) = \beta\alpha(c)$, $\alpha\beta(b) = \alpha\beta(c)$ i.e. α^2, β^2 and $\alpha\beta$ and $\beta\alpha$ are constant.

Hence the theorem is proved. ■

Theorem 6. *Let $(S, \cdot)$ be a weak commutative AG-groupoid. Define "$*$" as $a * b = (ap)b$, where p is fixed element in S. Then $(S, *)$ is commutative semigroup.*

Proof. Let $a, b, c \in S$. Then by definition of "$*$" and by left invertive law, we have $a * b = (ap)b = (bp)a = b * a$. Now by left invertive law and weak-commutativity, we have

$$(a * b) * c = (((ap)b)p)c \quad = \quad (cp)(ap \cdot b) = (cp)(bp \cdot a)$$

$$= (a(bp))(pc) = (ap)((bp)c) \quad = \quad a * (b * c) \longrightarrow (a * b) * c = a * (b * c).$$

Hence $(S, *)$ is commutative semigroup.

Enumeration of Weak Commutative AG-groupoids

Enumeration and classification of associative structures, semigroups and monoids have been done up to size 9 and 10 respectively by constraint satisfaction techniques implemented in the Minion constraint solver with bespoke symmetry breaking provided by the computer algebra system GAP. M. Shah and A. Distler (the author of (A. Distler et al.,(2010), A. Distler et al.,(2008), A. Distler et al.,(2009)) have enumerated AG-groupoids using a similar technique of semigroups and monoids.

It is worth mentioning that most of the data presented in (A. Distler et al.,(2010)) has been verified by one of the reviewers of the said article with the help of Mace-4 and Isofilter as has been mentioned in the acknowledgment of the said article. Using the same technique and data of (A. Distler et al.,(2010)), we develop a coding in GAP to enumerate our newly introduced subclass of weak commutative AG-groupoids up to order 6. Note that all AG-groupoids of order 2 or less are commutative and hence associative that are not of our interest.

Table 1 presents the enumeration of WC-AG-groupoids and its subclasses of order 3 to 6.

Order	3	4	5	6
Total AG-groupoids	20	331	31913	40104513
Non-associative AG-groupoids	8	269	31467	40097003
Total WC-AG-groupoids	18	313	31294	39960206
Non associative WC-AG-groupoids	6	251	30848	39952696
Associative but not commutative WC-AG-groupoid	0	4	121	5367
Commutative and associative WC-AG-groupoids	12	58	325	2143

Table 1. Enumeration and classification of WC-AG-groupoids up to order 6.

6. Congruences on weak-commutative AG-groupoids

Here, we define some congruences on weak-commutative AG-groupoids. We start with the following theorem.

Theorem 7. *For any a, b in a WC-AG-groupoid* $(S, \cdot)$, *let* η *be a relation defined on S as* $a\eta b$, *if for all* $x \in S$, $xa = xb$, *i.e.* $a\eta b \Leftrightarrow xa = xb$. *Then* η *is congruence on S.*

Proof. Let S be a WC-AG-groupoid. A relation η is defined on S as;

$$a\eta b \Leftrightarrow xa = xb, \text{ for } a, b \in S \text{ and } x \in S.$$

First we show that η is equivalence relation on S, for this we have to show that the relation η is reflexive, symmetric and transitive. For reflexive relation let for any $a \in S$ and for all $x \in S$, we have $xa = xa \longrightarrow a\eta a$. Hence η is reflexive. Again for any a, $b \in S$ and for all $x \in S$, let $a\eta b \Leftrightarrow xa = xb \Leftrightarrow xb = xa \Leftrightarrow b\eta a$. Hence η is symmetric. Now, for transitivity, let $a\eta b$ and $b\eta c$. Then $a\eta b \Leftrightarrow xa = xb$ and $b\eta c \Leftrightarrow xb = xc$ for all $x \in S$. This implies $xa = xb = xc \Leftrightarrow xa = xc \Leftrightarrow a\eta c$. Hence η is transitive. Therefore η is an equivalence relation on S.

η is right compatible: Let

$$\begin{aligned} a\eta b &\Leftrightarrow xa = xb \\ &\Leftrightarrow xa \cdot yc = xb \cdot yc \\ &\Leftrightarrow xy \cdot ac = xy \cdot bc \; a\eta b \Leftrightarrow ac\eta bc. \end{aligned}$$

Therefore, η is right compatible.

η is left compatible: Let

$$\begin{aligned} a\eta b &\Leftrightarrow xa = xb \\ &\Leftrightarrow (xa \cdot c)y = (xb \cdot c)y \\ &\Leftrightarrow yc \cdot xa = yc \cdot xb \\ &\Leftrightarrow yx \cdot ca = yx \cdot cb \; a\eta b \Leftrightarrow ca\eta cb. \end{aligned}$$

Therefore η is left compatible. Hence η is compatible and therefore η is congruence on S. ■

The following example depicts the above result.

Example 4. ***For*** $\forall x \in S$ ***the relation*** η ***given below is congruence for the following WC-AG-groupoid***

(S, .):	1	2	3	4	5

1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	1	1	1
4	2	2	1	2	2
5	2	2	3	2	3

$\eta = \{(1,1),(2,2),(3,3),(4,4),(5,5),(1,2),(2,1),(1,4),(4,1)\}$

Theorem 8. *Let S be a weak-commutative AG-groupoid, let $E(S) \neq \phi$ where $E(S)$ is the set of all idempotents of S and ρ be a relation defined on S as*

$$\rho = \{(a,b) \in S,\ ea = eb \text{ for every } e \in E(S)\}.$$

Then ρ is a congruence on S.

Proof. Let S be a WC-AG-groupoid and $E(S)$ denote the set of all idempotent elements in S. A relation ρ

is defined on S as,

$$\rho = \{(a,b) \in S,\ ea = eb \text{ for every } e \in E(S)\}.$$

First we show that ρ is an equivalence relation on S, for reflexive relation let for any $a \in S$ and $e \in E(S)$, we have $ea = ea \longrightarrow a\rho b$. Hence ρ is reflexive. Again let $a\rho b \Leftrightarrow ea = eb \Leftrightarrow eb = ea \Leftrightarrow b\rho a$. Hence ρ is symmetric. Now for transitivity, let $a\rho b$ and $b\rho c$, then $ea = eb,\ fb = fc$ for some $e, f \in E(S)$. Now

$$\begin{aligned}(ef)a &= (ee \cdot f)a = af \cdot ee = ee \cdot fa \\ &= ef \cdot ea = ef \cdot eb = ee \cdot fb \\ &= ee \cdot fc = cf \cdot ee = cf \cdot e = (ef)c \\ \longrightarrow (ef)a &= (ef)c\end{aligned}$$

Since $ef \in E(S)$ we conclude $a\rho c$ and ρ is transitive. Therefore ρ is an equivalence relation on $S.\rho$ is right compatible:

$$\begin{aligned}a\rho b &\Leftrightarrow ea = eb \\ &\Leftrightarrow ea \cdot c = eb \cdot c \\ &\Leftrightarrow ca \cdot e = cb \cdot e \\ &\Leftrightarrow ca \cdot ee = cb \cdot ee \\ &\Leftrightarrow ee \cdot ac = ee \cdot bc \\ &\Leftrightarrow e \cdot ac = e \cdot bc \\ a\rho b &\Leftrightarrow ac\rho bc.\end{aligned}$$

$$\begin{aligned}a\rho b &\Leftrightarrow ea = eb \\ &\Leftrightarrow ea \cdot c = eb \cdot c \\ &\Leftrightarrow ca \cdot e = cb \cdot e \\ &\Leftrightarrow ce \cdot ae = ce \cdot be \\ &\Leftrightarrow ea \cdot ec = eb \cdot ec\end{aligned}$$

ρ is left compatible:

Therefore, ρ is left and right compatible and hence is compatible. Equivalently ρ is congruence on S.■

Example 5. ***The relation ρ given below is congruence for the following WC-AG-groupoid.***

$\cdot$	1	2	3	4
1	1	1	3	3
2	1	1	4	4
3	3	3	1	1
4	3	3	1	1

In Example (5) for $e = 1$ the relation

$$\rho = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$$

is a congruence on S.

Theorem 9. *If S is an inverse WC-AG-groupoid then the relation,*

$$a \leq b \iff a = aa^{-1} \cdot b \tag{7.1}$$

on S is a partial order and is compatible.

Proof. Since S is an inverse WC-AG-groupoid, thus the relation $\leq$ is reflexive. Now for antisymmetric, assume $a \leq b$ and $b \leq a$, then $a = aa^{-1} \cdot b$ and $b = bb^{-1} \cdot a$. Thus

$$\begin{aligned} a = aa^{-1}\cdot b = (aa^{-1})(bb^{-1}\cdot a) &= (a\cdot bb^{-1})(a^{-1}a) = (aa^{-1})(bb^{-1}\cdot a) \\ = (aa^{-1})(ab^{-1}\cdot a) = (b\cdot ab^{-1})(aa^{-1}) &= (ba^{-1})(ab^{1}\cdot a) = (a\cdot ab^{-1})(a^{-1}b) \\ = (a^{-1}b\cdot ab^{-1})a = (a^{-1}a\cdot bb^{-1})a &= (b^{-1}b\cdot aa^{-1})a = (a\cdot aa^{-1})(b^{-1}b) \\ = (bb^{-1})(aa^{-1}\cdot a) = bb^{-1}a &= b. \end{aligned}$$

Thus $a = b$. Hence S is antisymmetric. For transitivity, assume that $a \le b$ and $b \le c$ this implies that

$a = aa^{-1}\cdot b$ and $b = bb^{-1}\cdot c$. So

$$\begin{aligned} a = aa^{-1}\cdot b = (aa^{-1})(bb^{-1}\cdot c) &= ((aa^{-1}\cdot a)a^{-1})(bb^{-1}\cdot c) \\ = (a^{-1}a\cdot aa^{-1})(bb^{-1}\cdot c) &= (a^{-1}a\cdot bb^{-1})(aa^{-1}\cdot c) \\ = ((bb^{-1}\cdot a)a^{-1})(aa^{-1}\cdot c) &= (a^{-1}a\cdot bb^{-1})(aa^{-1}\cdot c) \\ = (c\cdot aa^{-1})(bb^{-1}\cdot a^{-1}a) &= (c\cdot aa^{-1})(aa^{-1}\cdot b^{-1}b) \\ = (c\cdot aa^{-1})(ba^{-1}\cdot b^{-1}a) &= (c\cdot aa^{-1})((b^{-1}a\cdot a^{-1})b) \\ = (c\cdot aa^{-1})((ba^{-1}\cdot a)^{-1})b) &= (c\cdot aa^{-1})((aa^{-1}\cdot b)^{-1}b) \\ = (c\cdot aa^{-1})(a^{-1}b) = (ba^{-1})(aa^{-1}\cdot c) &= (ba^{-1})(ca^{-1}\cdot a) = (b\cdot ca^{-1})(a^{-1}a) \\ = (a\cdot ca^{-1})(a^{-1}b) = (aa^{-1})(ca^{-1}\cdot b) &= (aa^{-1})(ba^{-1}\cdot c) = (c\cdot ba^{-1})(a^{-1}a) \\ = (ca^{-1})(ba^{-1}\cdot a) = (ca^{-1})(aa^{-1}\cdot b) &= ca^{-1}\cdot a = aa^{-1}\cdot c \end{aligned}$$

Thus $a = aa^{-1}\cdot c$ or $a \le c$. Hence the relation $\le$ is transitive, so the relation $\le$ is a partial order on S.

Now for left compatibility, let $a \le b$ and $c \in S$. Then

$$\begin{aligned} ca = c(aa^{-1}\cdot b) = (cc^{-1}\cdot c)(aa^{-1}\cdot b) &= (b\cdot aa^{-1})(c\cdot cc^{-1}) = (bc)(aa^{-1}\cdot cc^{-1}) \\ = (bc)(ac\cdot a^{-1}c^{-1}) = (a^{-1}c^{-1}\cdot ac)(cb) &= (ca\cdot c^{-1}a^{-1})(cb) = (ca\cdot (ca)^{-1})cb \end{aligned}$$

Thus $ca = (ca\cdot (ca)^{-1})cb)$ or $ca \le cb$. Hence the relation $\le$ is left compatible. Also,

$$ac = (aa^{-1}\cdot b)c = (aa^{-1}\cdot b)(cc^{-1}\cdot c) = (aa^{-1}\cdot cc^{-1})(bc) =$$

$$= (ac\cdot a^{-1}c^{-1})bc = (ac\cdot (ac)^{-1})bc \longrightarrow ac \le bc.$$

Thus the relation $\le$ is right compatible, and hence compatible. ■

Corollary 2. *Let S be an inverse WC-AG-groupoid and $a, b \in S$. Then*

$$a \le b \Leftrightarrow aa^{-1} = ba^{-1}.$$

Proof. If $a \le b$ then by equation (7.1) we have

$$aa^{-1} = (aa^{-1}\cdot b)a^{-1} = ((aa^{-1})(bb^{-1}\cdot a))a^{-1}$$
$$= ((a\cdot bb^{-1})(a^{-1}a))a^{-1} = (a^{-1}\cdot a^{-1}a)(a\cdot bb^{-1})$$
$$= (bb^{-1}\cdot a)(a^{-1}a\cdot a^{-1})\ aa^{-1} = ba^{-1}$$

Conversely, for $a, b \in S$, $aa^{-1} = ba^{-1}$ implies that

$$a = aa^{-1}\cdot a = ba^{-1}\cdot a = aa^{-1}\cdot b.$$

So, by Equation (7.1), $a \leq b$. ■

Remark 1. *Let S be a WC-AG-groupoid and $e, f \in E(S)$. Then*

$$ef = ee\cdot ff = ff\cdot ee = fe.$$

This implies that $E(S)$ is a semi-lattice.

Remark 2. (M. Rashad et al.,(2016)) *Let S be an inverse AG-groupoid, $a \in S$, $a' \in V(a)$ and $aa' = a'a$. Then*

$$(aa')^2 = aa'\cdot aa' = (a'a\cdot a')a = a'a = aa'.$$

implies that $aa' \in E(S)$.

An inverse AG-groupoid, for element a, the product aa^{-1} and $a^{-1}a$ are not necessarily idempotent.

Lemma 7.0.1. *Let S be an inverse AG-groupoid, $a \in S$. Then $aa^{-1}, a^{-1}a \in E(S) \Leftrightarrow aa^{-1} = a^{-1}a$.*

Proof. Let $aa^{-1}, a^{-1}a \in E(S)$. Then, by left invertive law

$$aa^{-1}\cdot a^{-1}a = (a^{-1}a\cdot a^{-1})a = a^{-1}a.\ a^{-1}a\cdot aa^{-1} = (aa^{-1}\cdot a)a^{-1} = aa^{-1}.$$

but $aa^{-1}, a^{-1}a \in E(S)$ and by Remarks (1) $E(S)$ is semi-lattice, that is $aa^{-1}\cdot a^{-1}a = a^{-1}a\cdot aa^{-1}$. So the result follows from above that is $aa^{-1} = a^{-1}a$.

Converse follows easily from Remarks (2).■

Example 6. *Inverse AG-groupoid S of order 4, be given below*

*	1	2	3	4
1	2	2	4	4
2	2	2	2	2
3	1	2	3	4
4	1	2	1	2

Then $E(S) = \{2, 3\}$ is a semi-lattice, elements 1 and 4 are mutually inverses and $1 * 4 \neq 4 * 1$.

Corollary 3. *Let $a, b \in S$. Where S is completely inverse WC-AG-groupoid. Then*

$$a \leq b \Leftrightarrow (\exists e \in E(S))\, a = eb.$$

Proof. Let $a, b \in S$. Where S is completely inverse WC-AG-groupoid. Then $a \leq b$ if and only if $a = aa^{-1} \cdot b$.

Since $aa^{-1} \in E(S)$, therefore if $aa^{-1} = e$ implies that $a = eb$.

Conversely, let $a, b \in S$, be such that $e \in E(S)$ and $a = eb$ because $aa^{-1} = a^{-1}a \in E(S)$ and $E(S)$ is semi-lattice, we have

$$aa^{-1} \cdot b = eb \cdot eb^{-1}\ b = ee \cdot bb^{-1}\ b = b \cdot bb^{-1}\ (ee)$$

$$= (ee)\ bb^{-1} \cdot b = eb = a$$

and so, $a \leq b$. ■

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